Analytical Expressions for the Gravitational Constant

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1. Introduction

Theories which unify gravity (the Einstein equations of general relativity) and electromagnetism (the Maxwell equations of electrodynamics and/or quantum electrodynamics) can be expected to provide analytical expressions of Newton's gravitational constant *G* in terms of other fundamental parameters. Indeed, one may naively expect that a successful analytical prediction may even validate the underlying theory. Yet it turns out that there is a fairly rich history of analytical expressions for the gravitational constant and no obvious way to choose between them. This article introduces the subject by considering a compelling expression for *G* developed relatively recently by Brandenburg, and then considering the expression in the context of the general history of some previous expressions for *G*.

2. Brandenburg expression for *G*

Brandenburg (1992) provided an expression for *G* in terms of the quantum of electric charge *q*, the electron mass m_e, the proton mass m_p, and the fine structure constant α (the latter constructed from the Planck constant and the speed of light):

$$G_B = \frac{\alpha q^2}{m_p m_e} e^{-2(m_p/m_e)^{1/2}}$$

in cgs units. Brandenburg's expression followed from considerations of the unification of gravity and quantum electrodynamics.

NIST provides the CODATA values (2006) of the fundamental constants (converted to cgs units and written to five significant figures):

$$\alpha = 7.2974\text{x}10^{-3}$$
$$q = 4.8032\text{x}10^{-10}$$
$$m_p = 1.6726\text{x}10^{-24}$$
$$m_e = 9.1094\text{x}10^{-28}$$

Using the CODATA 2006 values, Brandenburg calculates:

$$G_B = 6.6724\text{x}10^{-8}$$

The gravitational constant is the most poorly constrained of the physical constants. Whereas the fine structure constant uncertainty is below a part in a billion, the uncertainty of the gravitational constant is a part in ten thousand. The CODATA 2006 value is:

$$G = 6.674\text{x}10^{-8}$$

Brandenburg's value is within 0.03% of the measured value. This seems all the more dramatic when one notices the extreme sensitivity in the exponential term of Brandenburg's expression. On the face of it, this would seem to be as profound a prediction as when the Rydberg constant was described in 1913 in terms of these same physical constants. The immediate implication would be that *G* must be understood as a quantum electrodynamic effect. Let us now consider the Brandenburg expression in the context of some previous expressions for *G*.

3. *G* and the Large Number Hypothesis

Attempts to express *G* in terms of atomic parameters go back to P.A.M. Dirac and his Large Numbers Hypothesis of 1937. Dirac noted that the ratio of electrostatic to gravitational forces between an electron and a proton is a very large number approximately equal to the ratio of the size of the observable universe to the classical electron radius. In terms of the speed of light *c*, the age of the universe T_U, and the classical electron radius $r_e \equiv q^2/m_e c^2$:

$$\frac{c T_U}{r_e} \sim 10^{40} \sim \frac{q^2}{G m_p m_e}$$

If there were some cosmic influence underlying the equation of these ratios, and the atomic parameters were fixed, then Dirac concluded that $G \propto 1/T_U$, thus changing as the universe evolves. [Note, however, that standard big bang cosmology and observation constrain *G* to be relatively unvarying over the age of the universe. See, e.g., the review by Uzan (2003).]

Dirac's expression is solved for *G* :

$$G_D = \frac{q^4}{m_e m_p} \cdot \frac{1}{T_U m_e c^3}$$

Returning to Brandenburg's expression, it is expanded:

$$G_B = \frac{q^4}{m_e m_p} \cdot \frac{e^{-2(m_p/m_e)^{1/2}}}{\hbar c}$$

where $\hbar$ is the Planck constant. When the expressions of Brandenburg and Dirac are combined, one obtains:

$$cT_{U,BD} = \frac{\hbar}{m_e c}\, e^{2(m_p/m_e)^{1/2}}$$

This expression for the size of the universe is in terms of the Compton wavelength of the electron multiplied by an exponential inflation factor that depends on the electron and proton masses. It is remarkable that the universal inflation scale follows from the proton-electron mass ratio.

4. *G* and quantum field theory

Another influential insight into the nature of *G* was provided by Sakharov (1967) who proposed interpreting *G* in terms of an effect of quantum field theory, viz., in terms of an integral over the spectrum of electromagnetic zero point vacuum fluctuations, with the short wavelength cutoff given by the Planck length:

$$r_p \equiv \sqrt{G\hbar/c^3} \simeq 10^{-33}\text{cm} \quad .$$

Thus gravity is seen to be a quantum electrodynamic effect. However, Sakharov did not “predict” *G* since *G* was built into the integration limit through r_P.

Similar to Sakharov's expression for *G*, Brandenburg's expression G_B was obtained in terms of the Planck length. Brandenburg defined a “mesoscale” mass from a geometric mean of the proton and electron masses:

$$m_0 \equiv (m_p m_e)^{1/2} \quad ,$$

and a corresponding “classical mesoscale” radius,

$$r_0 \equiv q^2/m_0 c^2 \quad .$$

From these quantities Brandenburg posits the fundamental equality,

$$\left(m_p/m_e\right)^{1/2} = \ln\left(r_0/r_p\right) \quad ,$$

from which G_B is obtained. [J. Brandenburg, personal communication]

There is a compelling elegance to Brandenburg's fundamental equality, as well as in the expression for G_B. Yet the motivation for either equation does not flow from a set of intuitive assumptions as easily as did Bohr's expression for the Rydberg constant. Can the expression for G_B then be validated in terms of its accuracy? This is also problematic.

5. Other theoretical and numerical expressions for *G*

Gillies (1997) provides a review of theoretical values of *G*. Many of them are within the same 0.1% accuracy as the Brandenburg result. The gravitational constant is just too poorly known to distinguish from among theoretical values. Indeed, theorists often avail themselves of any degrees of freedom necessary to be consistent with observation to any desired degree of accuracy. For example, Krat and Gerlovin (1974) provide an expression with many factors:

$$G_{KG} = (1.000888)\frac{9}{32\pi^2}(\lambda_p R_\infty)^4 \frac{q^2}{m_p^2} \quad .$$

So perhaps one could distinguish between theoretical expressions on the basis of elegance, but this would be difficult to assess objectively. For example, the Brandenburg expression is more elegant than the Krat and Gerlovin expression. Yet even if we could find an objective measure of elegance, we cannot safely assume that nature will prefer an elegant *G*.

Even without a compelling physical argument or set of axioms underlying an expression for *G*, one finds some numerical expressions. Gillies (1997) describes how Landau suggested that:

$$G_L m_?^2/\hbar c = A e^{-B/\alpha} \quad ,$$

where $m_?$ is "the mass of a fundamental particle" and *A* and *B* are of order unity. Damour chose values of $m_?$, *A,* and *B*:

$$G_{LD} m_e^2/\hbar c = \frac{(7\pi)^2}{5} e^{-\pi/4\alpha} \quad .$$

(See Gillies (1997) for the citations to Landau and Damour).

Using the CODATA values (2006) for the Planck constant and speed of light (converted to cgs units and written to 5 significant figures),

$$\hbar = 1.0546\text{x}10^{-27}$$
$$c = 2.9979\text{x}10^{10} \quad ,$$

the Landau-Damour value is:

$$G_{LD} = 6.6771\text{x}10^{-8} \quad .$$

A purely numerical formula for G in terms of simple numbers (source unknown) is:

$$G_n = \frac{\pi}{e\sqrt{3}} \text{x}10^{-7} = 6.6726\text{x}10^{-8} \quad .$$

Indeed, Lenz (1951) noted that the mass ratio

$$m_p/m_e \simeq 6\pi^5 \quad .$$

The CODATA (2006) value for the mass ratio is (to seven significant figures):

$$m_p/m_e = 1836.153$$

whereas

$$6\pi^5 = 1836.118$$

This implies a numerical form for the Brandenburg expression:

$$G_B = \frac{q^4}{m_e m_p} \cdot \frac{e^{-2(6\pi^5)^{1/2}}}{\hbar c} \quad ,$$

a hybrid of the Dirac and Landau forms.

6. Conclusions

A theory which unifies gravity and electromagnetism is expected to provide a theoretical or analytical value for G which depends on electrodynamic and/or atomic constants. However, because such expressions are not unique, and because many such expressions can come within the accuracy to which *G* is measured, an accurate expression of this sort would only be a necessary condition, not a sufficient condition, to prove the validity of the underlying theory.

7. References

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