Thermal, Emissive, and Reflective Behavior of Spheres in Earth and Sun Radiation Fields

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Intro & Summary of Results

For comparison with and validation of optical signature modelling tools, this report calculates 4 optical signature components and the thermal cooling temperature profile of a sphere outside the earth atmosphere. The sphere is characterized by radius r and emissivity $\epsilon(\lambda, \psi)$, viewed over some waveband. The emissivity, depending on both direction relative to the surface normal and wavelength, is the key quantity characterizing both the emissive and reflective properties of a surface. The waveband will be assumed narrow enough that ϵ is approximately constant over the band.

Because of the symmetry of the target, the thermal radiant intensity will be independent of aspect angle, assuming a uniform temperature over the sphere. The reflected solar radiant intensity will depend only on the aspect angle ξ with respect to the sun-sphere line. Reflected earthshine radiant intensity can have a complicated angular dependence due to a non-uniform earth source function, as can the reflected earth albedo. Calculated radiant intensities of these components, in units of watts per steradian, are summarized below:

- Thermal Emission:

$$J_T \simeq 2r^2 g(T, \Delta\lambda) \int_0^{\pi/2} \epsilon(\psi) \sin\psi \cos\psi d\psi$$

- Reflected Sunshine
  - Specular:

$$J_{\odot}(\xi) = (1 - \epsilon(\xi/2)) F_{\odot} \frac{r^2}{4}$$

  - Diffuse:

$$\pi J_{\odot}(0) = 2\pi F_{\odot} r^2 \int_0^{\pi/2} (1 - \epsilon(\theta)) \cos^2\theta \sin\theta d\theta$$

$$\pi J_{\odot}(\xi < \pi/2) = F_{\odot} r^2 (2\pi \cos\xi I_1(\xi, \epsilon) + 2\sin\xi I_2(\xi, \epsilon) + 2\cos\xi I_3(\xi, \epsilon))$$

$$\pi J_{\odot}(\pi/2) = 2F_{\odot} r^2 \int_0^{\pi/2} (1 - \epsilon(\theta)) \cos\theta \sin^2\theta d\theta$$

$$\pi J_{\odot}(\xi > \pi/2) = F_{\odot} r^2 (2\sin\xi I_2(\xi, \epsilon) + 2\cos\xi I_3(\xi, \epsilon))$$

$$J_{\odot}(\pi) = 0$$

- Specularly Reflected Earthshine:

$$J_{\oplus}(\xi) = \frac{R_{\oplus}^2 r^2}{8\pi} g(T_{\oplus}, \Delta\lambda) \int_{R_{\oplus}/D}^{1} d\eta \; \epsilon_{\oplus}(\eta) \frac{(D\eta - R_{\oplus})(1 - \epsilon[\xi, \eta])}{(R_{\oplus}^2 + D^2 - 2R_{\oplus}D\eta)^{3/2}}$$

- Specularly Reflected Earth Albedo Estimate (sunshine reflected from earth onto sphere):

$$J_{\odot\oplus} \sim F_{\odot}(1 - \bar{\epsilon}_{\oplus})(1 - \bar{\epsilon}) \frac{R_{\oplus}^2}{l^2} \frac{r^2}{64\pi}$$

For a sphere cooling from temperature T_0 in the absence of any heating sources, the temperature profiles for 3 choices of emissivity are given below:

- Blackbody:

$$\frac{T}{T_0} = \left(\frac{t}{\tau_{bb}} + 1 \right)^{-1/3}$$

- Electron Gas:

$$\frac{T}{T_0} = \left(\frac{t}{\tau_{eg}} + 1 \right)^{-2/7}$$

- Numerical Emissivity:

$$t = h\rho d \sum_{T_j} \frac{\Delta T}{\sum_{\lambda_i} f(T_j, \lambda_i)\epsilon(\lambda_i)\Delta\lambda}$$

The blackbody and electron gas profiles represent perfect emitters and perfect mirrors, respectively, and are expected to bracket the behavior of any real material.

THERMAL EMISSION

In terms of the emissivity $\epsilon(\psi, \lambda)$, the spectral radiance of an element of surface area da at temperature T is given by:

$$N(\lambda, \psi) = \epsilon(\lambda, \psi) \frac{\cos\psi}{\pi} f(T, \lambda) \qquad (\mathrm{W \cdot cm^{-2} \mu m^{-1} sr^{-1}})$$

where f is the blackbody ($\epsilon = 1$) spectral radiant emittance:

$$f(T, \lambda) = \frac{c_1}{\lambda^5} \left[\mathrm{e}^{c_2/\lambda \mathrm{T}} - 1 \right]^{-1}$$

$$c_1 = 3.74 \times 10^4 \ \ \mathrm{W \cdot cm^{-2} \mu m^4} \qquad c_2 = 1.44 \times 10^4 \ \ \mathrm{\mu m \cdot K}$$

Assuming the temperature is uniform over the sphere, then, due to symmetry, the thermal radiant intensity J_T of the sphere can be simply obtained by integrating N over a hemisphere for each da, multiplying by the surface area $4\pi r^2$, and dividing by the total steradians, 4π:

$$\begin{aligned} J_T &= \frac{1}{4\pi} \int da \int_0^{\pi/2} \sin\psi \frac{\cos\psi}{\pi} d\psi \int_0^{2\pi} d\phi \int_{\lambda_1}^{\lambda_2} \epsilon(\psi, \lambda) f(T, \lambda) d\lambda \\ &\simeq 2r^2 g(T, \Delta\lambda) \int_0^{\pi/2} \epsilon(\psi) \sin\psi \cos\psi d\psi \qquad (\mathrm{W \cdot sr^{-1}}), \qquad \epsilon(\lambda_1) \simeq \epsilon(\lambda_2) \end{aligned}$$

where $g(T, \Delta\lambda)$ is the integral of $f(T, \lambda)$ over the waveband of interest. The assumption was made that the emissivity is constant over the waveband of interest. This is a good approximation for wavebands narrower than a micron or so.

TEMPERATURE PROFILE OF EMISSIVE COOLING

Consider the change in temperature of a sphere cooling via thermal emission in the absence of any outside heating source. The total power $R(T)$ emitted at temperature T is obtained by integrating the emission over all solid angle for an element of surface area, and then multiplying by $4\pi r^2$. Unlike the previous treatment of thermal emission over some small waveband, it is necessary to integrate over all wavelengths (although the bulk of the emission will indeed come from a relatively small waveband). This requires allowance for a dependence of emissivity on wavelength.

$$R(T) = 4\pi r^2 \int_0^\infty f(T,\lambda)d\lambda \int_0^{2\pi} d\phi \int_0^{\pi/2} \epsilon(\theta,\lambda)\frac{\cos\theta}{\pi}\sin\theta d\theta \qquad \text{(Watts)}$$

For integration over all wavelengths, real emissivities are found to vary more in wavelength than angle. Since most of the emission from an element of surface area will be into a relatively small cone of solid angle about the normal direction, a good approximation is to ingnore the angular dependence of ϵ. An appropriate estimate for $\epsilon(\lambda)$ is its normal direction value; the cosine factor kills off tangential emission.

A little calculus can relate R to the heat capacity H:

$$\begin{aligned} R(T) &\equiv \frac{dE}{dt} && \text{energy per time} \\ &= \frac{dE}{dT}\frac{dT}{dt} \\ &\equiv H\frac{dT}{dt} \end{aligned}$$

so that the time t to cool from temperature T_0 to T is given by:

$$\int_{T_0}^{T} \frac{H}{R} dT = \int_0^t dt = t$$

In terms of the specific heat capacity h, mass density ρ, and thickness d of the sphere (assume $d \ll r$), $H = h4\pi r^2 d\rho$. Although h is in general a function of temperature, for relatively small temperature intervals, say less than 100 K, it can be taken constant.

- Case: Blackbody

The emissivity $\epsilon = 1$ for all wavelengths, so that the angular integral collapses to 1, and the spectral integral is just:

$$\int_0^\infty f(T, \lambda) d\lambda = \sigma T^4$$

the Stephan-Boltzmann law, where $\sigma = 5.67 \times 10^{-8} \mathrm{W/m^2K^4}$.

The time to cool from T_0 to T, at constant h, is then:

$$t = \frac{h\rho d}{3\sigma} \left(\frac{1}{T^3} - \frac{1}{T_0^3} \right)$$

Invert this to obtain $T(t)$:

$$\frac{T}{T_0} = \left(\frac{t}{\tau_{bb}} + 1 \right)^{-1/3} \qquad \tau_{bb} = \frac{h\rho d}{3\sigma T_0^3}$$

At $t = \tau_{bb}$, $T = 0.8T_0$. Using the heat capacity and density of carbon-carbon, a material which may be expected to have emissive properties similar to a blackbody, the cooling timescale $\tau_{bb} = 1.4$ hr, for a thickness of 2 cm. For behavior on very short timescales,

$$\frac{T}{T_0} \simeq 1 - \frac{1}{3} \frac{t}{\tau_{bb}}, \qquad t \ll \tau_{bb}$$

- Case: Electron Gas

This is the case of an idealized metal, in which the optical properties are determined only by the conduction electrons, which can be treated mathematically as an electron gas within the lattice of ions. Refer to the previous report "The Optics of Metals", for derivation of the optical properties of an electron gas. This approximation is valid for frequencies at about the visible and below. It forces a restriction to temperatures below about 5000 K; the Planck function kills off contributions from frequencies where this emissivity is invalid. In terms of the conductivity κ and angular radiation frequency ω:

$$\epsilon_{eg} \simeq 2\sqrt{\frac{\epsilon_0 \omega}{\kappa}}$$

Here, ϵ_0 is the standard notation for the permittivity of free space; do not confuse it with an emissivity.

ϵ_{eg} is the value of emissivity in the surface normal direction. There is a weak angular dependence (emissivity *increases* slightly away from normal), but a good approximation is to take this value as constant for all angles. Then:

$$\int_0^\infty \epsilon_{eg}(\lambda) f(T,\lambda) d\lambda = 3.8\, \sigma \sqrt{\frac{k\epsilon_0}{\kappa h}}\, T^{9/2} \equiv \Sigma\, T^{9/2}$$

where k is the Boltzmann constant.

The time to cool from T_0 to T is then:

$$t = \frac{2h\rho d}{7\Sigma}\left(\frac{1}{T^{7/2}} - \frac{1}{T_0^{7/2}}\right)$$

Invert this to obtain $T(t)$:

$$\frac{T}{T_0} = \left(\frac{t}{\tau_{eg}} + 1\right)^{-2/7}, \qquad \tau_{eg} \equiv \frac{2h\rho d}{7\Sigma T_0^{7/2}}$$

At $t = \tau_{eg}$, $T = 0.82T_0$. Using the heat capacity and density of stainless steel, a material which may be expected to have emissive properties similar to an electron gas, the cooling timescale $\tau_{eg} = 58$ hr, for a thickness of 2 cm. For behavior on very short timescales,

$$\frac{T}{T_0} \simeq 1 - \frac{2}{7}\frac{t}{\tau_{eg}}, \qquad t \ll \tau_{eg}$$

- Case: Numerical ϵ

The emissivities of real materials are typically in a tabular form over angle and wavelength, albeit usually at only a few discrete values. Since the above derived expressions involve integrals of integrals, the discrete sums are recast here.

$$\frac{H}{R} \Longrightarrow \frac{h\rho d}{\sum_{\lambda_i} f(T,\lambda_i)\epsilon(\lambda_i)\Delta\lambda} \equiv \frac{h\rho d}{Q(T)}$$

A vector Q_j is constructed by evaluating Q at each T_j. Then the time integral becomes:

$$t = h\rho d \sum_{T_j} \frac{\Delta T}{Q(T_j)}$$

REFLECTED SUNSHINE

A great simplification to reflection calculations arises for sources at infinity, so that the incident rays are approximately parallel. The sun can be well-approximated as such a source. The solar source spectrum can be approximated as a blackbody $f(T_\odot, \lambda)$. The solar irradiance at earth orbit in the waveband of interest is then:

$$F_\odot = \left(\frac{R_\odot}{1 \text{ AU}}\right)^2 \int_{\lambda_1}^{\lambda_2} f(T_\odot, \lambda) d\lambda$$

Reflectivity can be modelled as a linear combination of pure diffuse and pure specular. These limiting cases are considered separately below.

Specular Reflection

Refer to Figure 1 for an illustration of the geometry. Consider an incident solar flux of $F_\odot$ W/cm^2, and restrict attention to rays in the annulus of width db about impact parameter b. Rays not absorbed are specularly scattered into an annulus of solid angle of width $d\xi$ about angle ξ from the sun direction. Equating these fluxes in terms of the solar specular radiant intensity $J_\odot$ and the reflectivity $= 1 - \epsilon$:

$$(1 - \epsilon(\alpha))F_\odot 2\pi b db = J_\odot(\xi) 2\pi \sin\xi d\xi$$

Since the angle of incidence = the angle of reflection, the angle of the incident radiation with respect to the surface normal $\alpha = \xi/2$. Since $bdb = r^2 \sin(\xi/2)\cos(\xi/2)d\xi/2$,

$$J_\odot(\xi) = (1 - \epsilon(\xi/2))F_\odot \frac{r^2}{4} \qquad \text{(W/sr)}$$

Should depend on ξ? No!

Diffuse Reflection

Consider now pure diffuse reflection. Then, the flux incident on da that is not absorbed is 'reflected' into solid angle proportional to the cosine of the surface normal. Such reflective (or emissive) behavior is also called 'Lambertian', and is obtained when $\epsilon = 1$. Here, we make use of $\epsilon \neq 1$ to account for fractional absorption, but the redistributed radiation is that of a blackbody emitter. Physically, diffuse reflection arises not from the emissivity of a material, but from surface roughness. IR reflections tend to be specular, but as the radiation wavelength approaches the scale of irregularities in the surface, the reflection becomes more diffuse. Visible reflectivities may have an appreciable diffuse component. Refer to Figure 2 for an illustration of the geometry.

Consider now the target sphere at the origin of coordinates, with the incident solar flux along $-\hat{z}$, and the observation direction at polar angle ξ in the $\hat{x} - \hat{z}$ plane. Each da is at polar coordinates θ, ϕ. As for the specular case, the total reflected radiation by each da is

$$dF = (1 - \epsilon(\theta))F_{\odot} b db d\phi \qquad \text{(W)}$$

The density of this radiation in solid angle is the diffuse reflected radiant intensity by da:

$$dJ_{\odot}(\xi) = \frac{\cos\psi}{\pi} dF \qquad (\text{W} \cdot \text{sr}^{-1})$$

where ψ is the angle between the observer direction and the surface normal.

The normal direction of each da is

$$\hat{n} = \cos\phi \sin\theta \hat{x} + \sin\phi \sin\theta \hat{y} + \cos\theta \hat{z}$$

and the observer direction is:

$$\hat{O} = \sin\xi \hat{x} + \cos\xi \hat{z}$$

so that:

$$\cos\psi = \cos\phi \sin\theta \sin\xi + \cos\theta \cos\xi$$

Since $b db = r^2 \sin\theta \cos\theta d\theta$, the radiation scattered into direction ξ by $da(\theta, \phi)$:

$$dJ_{\odot}(\xi) = \frac{r^2 F_{\odot}}{\pi}(1 - \epsilon(\theta))(\cos\phi \sin\theta \sin\xi + \cos\theta \cos\xi) \sin\theta \cos\theta d\theta d\phi$$

This expression is integrated over the visible illuminated region to yield the total reflected radiant intensity at ξ.

The boundary of the visible region is defined by $\cos\psi = 0$. This can be viewed as the function $\phi_b(\theta)$, which defines the integration limit in ϕ for each value of θ:

$$\cos\phi_b = -\cot\xi\cot\theta$$

The region of ϕ integration is then $-\phi_b \leq \phi \leq \phi_b$. The region of θ integration is $0 \leq \theta \leq \pi/2$, the illumination region, for all ϕ. The integral simplifies for two special cases: $\xi = 0, \pi/2$.

- Case: $\xi = 0$

Here, the visible region is just the illuminated region. Then, $\cos\psi = \cos\theta$, so that:

$$\frac{\pi J_\odot(0)}{F_\odot r^2} = 2\pi\int_0^{\pi/2}(1-\epsilon(\theta))\cos^2\theta\sin\theta d\theta$$

- Case: $\xi = \pi/2$

Here, the visible region coincides with $-\pi/2 \leq \phi \leq \pi/2$, and $\cos\psi = \cos\phi\sin\theta$.

$$\begin{aligned}\frac{\pi J_\odot(\pi/2)}{F_\odot r^2} &= \int_0^{\pi/2}(1-\epsilon(\theta))\cos\theta\sin^2\theta d\theta\int_{-\pi/2}^{\pi/2}\cos\phi d\phi \\ &= 2\int_0^{\pi/2}(1-\epsilon(\theta))\cos\theta\sin^2\theta d\theta\end{aligned}$$

- Case: $0 < \xi < \pi/2$

It turns out that the expression for ϕ_b must be carefully applied; for a given ξ, values of θ too small or large yield unphysical imaginary ϕ_b (see figures). By plotting the imaginary part of ϕ_b, one can identify an interval in θ of real ϕ_b, bounded by θ_{min} and θ_{max}. By induction from such plots, $\theta_{min} = \pi/2 - \xi$, and $\theta_{max} = \pi/2 + \xi$. Values of $\theta < \theta_{min}$ are those latitudes that are illuminated and visible for all ϕ. The boundary ϕ_b turns on for $\theta > \theta_{min}$. The upper θ limit is just $\pi/2$, the limit of the illuminated region, since $\theta_{max} > \pi/2$.

$$\begin{aligned}
\frac{\pi J_\odot(\xi < \pi/2)}{F_\odot r^2} &= \int_0^{\theta_{min}} (1 - \epsilon(\theta)) \cos\theta \sin\theta d\theta \int_0^{2\pi} (\cos\phi \sin\theta \sin\xi + \cos\theta \cos\xi) d\phi \\
&\quad + \int_{\theta_{min}}^{\pi/2} (1 - \epsilon(\theta)) \cos\theta \sin\theta d\theta \int_{-\phi_b}^{\phi_b} (\cos\phi \sin\theta \sin\xi + \cos\theta \cos\xi) d\phi \\
&= 2\pi \cos\xi \int_0^{\theta_{min}} (1 - \epsilon) \cos^2\theta \sin\theta d\theta \\
&\quad + 2 \sin\xi \int_{\theta_{min}}^{\pi/2} (1 - \epsilon) \cos\theta \sin^2\theta \sin\phi_b d\theta \\
&\quad + 2 \cos\xi \int_{\theta_{min}}^{\pi/2} (1 - \epsilon) \cos^2\theta \sin\theta \phi_b d\theta \\
&\equiv 2\pi \cos\xi I_1(\xi, \epsilon) + 2 \sin\xi I_2(\xi, \epsilon) + 2 \cos\xi I_3(\xi, \epsilon)
\end{aligned}$$

- Case: $\pi/2 < \xi < \pi$

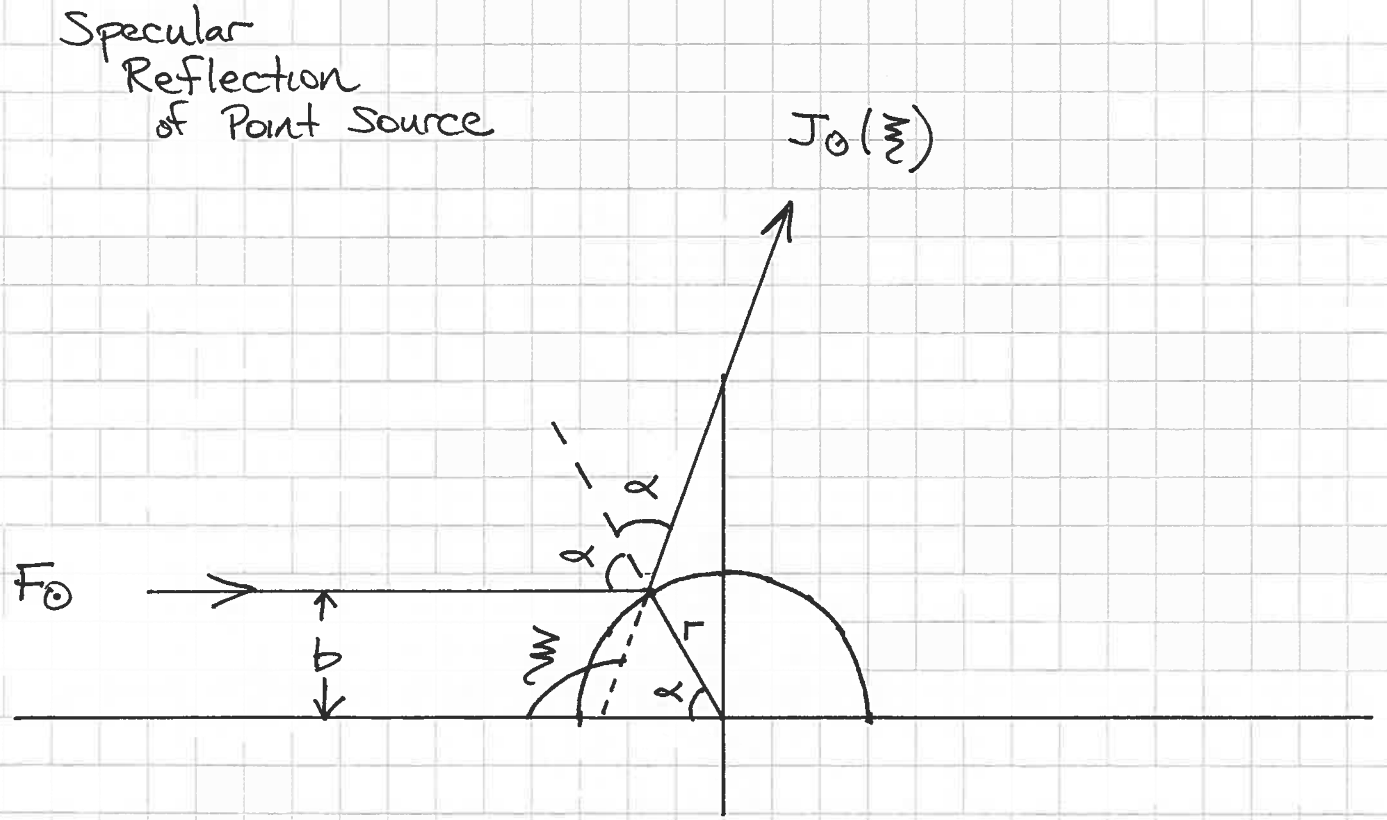
The difference in calculation from the previous is the treatment of values of $\theta < \theta_{min}$. As previously, a naive application of the expression for ϕ_b can yield imaginary values. And as previous, graphically displaying ϕ_b indicates the real values are contained within a range of θ. However, this time $\theta_{min} = \xi - \pi/2$ and $\theta_{max} = 3\pi/2 - \xi$. Unlike the previous case, θ_{min} defines the boundary above which the illuminated portion is not visible. The upper boundary of θ is still defined by $\pi/2$, the illumination boundary.

$$\frac{\pi J_\odot(\xi > \pi/2)}{F_\odot r^2} = 2 \sin\xi I_2(\xi, \epsilon) + 2 \cos\xi I_3(\xi, \epsilon)$$

- Finally, $J_\odot(\pi) = 0$, since none of the illuminated region is visible.
- A useful check on the expression for $J_\odot(\xi)$ can be obtained by equating the total flux reflected to the sum of $J_\odot(\xi)$ over solid angle:

$$F_\odot r^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} (1 - \epsilon(\theta)) \sin\theta \cos\theta d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} J_\odot(\xi) \sin\xi d\xi$$

Figure 1



Diffuse Reflection

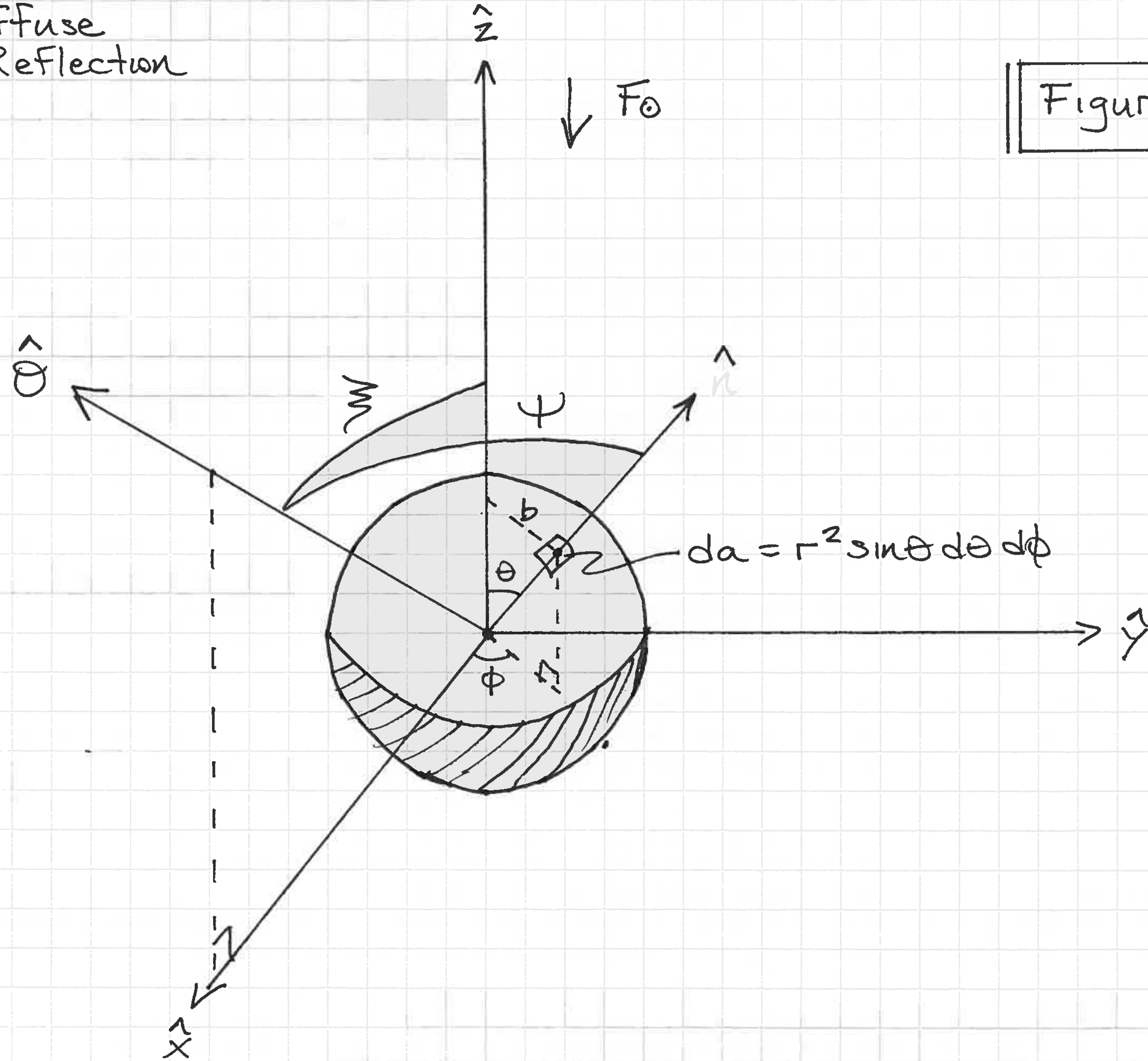
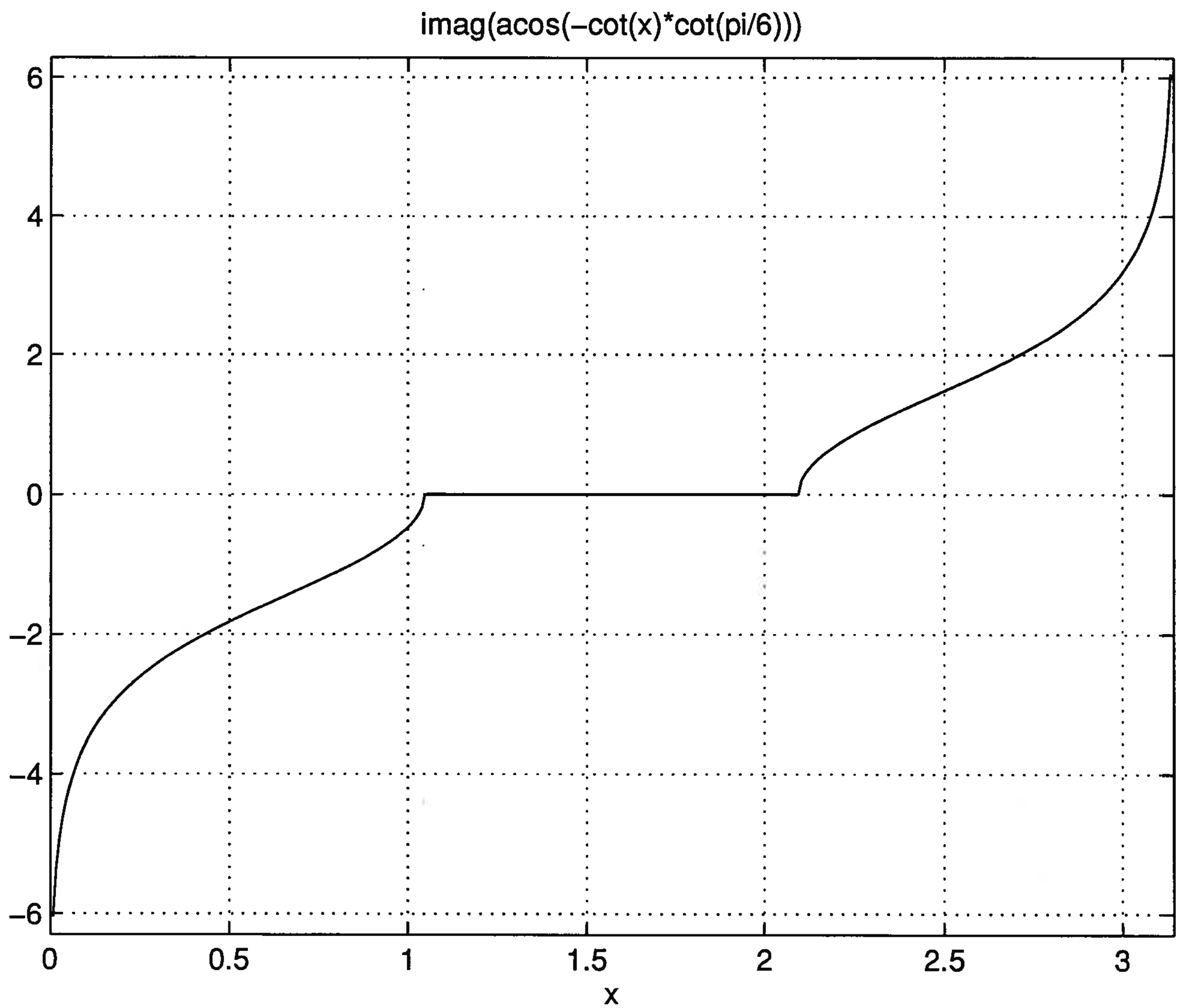
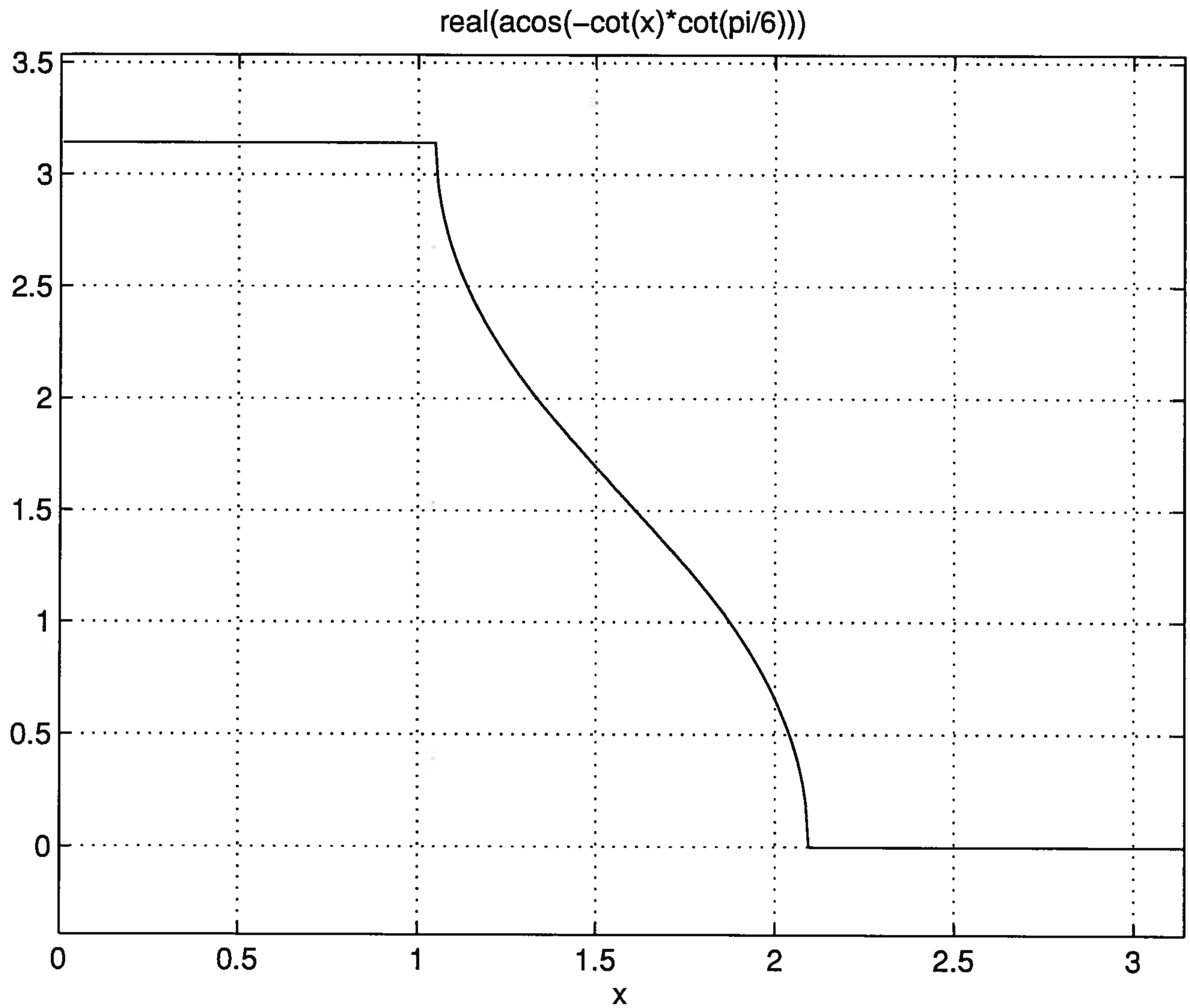


Figure 2

$\text{Im}\, \phi_b(x) \qquad (\xi = \pi/6)$



$\mathrm{Re}\,\phi_b(x) \qquad (\xi = \pi/6)$



SPECULARLY REFLECTED EARTHSHINE

Calculations of this radiant intensity are complicated on two fronts: the finite extent of the earth with respect to the sphere, and the generally non-uniform temperature and emissivity of the earth. The former is only a mathematical complication, and while not as simple as a point source treatment, is still tractable. But the latter causes the radiant intensity to depend sensitively on whatever necessarily-simplified assumptions are made about the earth source. The earth emission will vary not only spatially over the earth, but with time as well, due to weather and climate effects. The earth source function will be numerous absorption and emission lines superposed on the ~ 300 K blackbody spectrum. Various numerical models may use different tabulated earth source data, or a single model may have alternative sources for different times of year. The following treatment gives first a simple estimate for a point source earth, and then an exact calculation, all assuming specular reflectivity.

But first, consider the illumination of the target sphere. For a source at infinity, exactly one half the sphere is illuminated. But for an extended source, the illumination is greater than one half. And the illuminating source region is less than one half the earth. The latitude γ of the illuminated part of the sphere is related simply to the colatitude ϕ of the illuminating part of the earth: $\phi + \gamma = \pi/2$, and $\cos\phi \simeq R_{\oplus}/D$, where $R_{\oplus}$ is the radius of the earth and D is the distance to the sphere from the earth center. See Figure 3 for geometry. For $D = 2R_{\oplus}$, $\phi = 60°$.

Point Source Estimate

Consider now the specular reflected radiant intensity of a point source earth. In terms of the incident earth flux $F_{\oplus}$, the reflected radiant intensity is as given previously for solar radiation:

$$J_{\oplus}(\xi) = (1 - \epsilon(\xi/2))F_{\oplus}\frac{r^2}{4} \qquad \text{(W/sr)}$$

where ξ is now measured with respect to the earth-sphere line. Within the point source approximation, there are a couple possibilities for $F_\oplus$.

- Case: Blackbody earth

If the earth emissivity $\epsilon_\oplus = 1$, the earth source spectrum is a simple blackbody attenuated by distance:

$$F_\oplus = \left(\frac{R_\oplus}{D}\right)^2 \int_{\lambda_1}^{\lambda_2} f(T_\oplus, \lambda) d\lambda$$

- Case: Uniform earth

Here, $\epsilon_\oplus \neq 1$, but we assume $\epsilon_\oplus$ uniform over the earth surface (or better, a surface average). Then integrate the emission from each earth surface element $dA_\oplus$ over the hemisphere, just as for the thermal emission calculation above. As before, we make the reasonable assumption of constant emissivity over the waveband of interest.

$$\begin{aligned} F_\oplus &= \left(\frac{R_\oplus}{D}\right)^2 \int_{\lambda_1}^{\lambda_2} f(T_\oplus, \lambda) d\lambda \int_0^{2\pi} d\phi \int_0^{\pi/2} \epsilon_\oplus(\theta) \frac{\cos\theta}{\pi} \sin\theta d\theta \\ &= 2\left(\frac{R_\oplus}{D}\right)^2 \int_{\lambda_1}^{\lambda_2} f(T_\oplus, \lambda) d\lambda \int_0^{\pi/2} \epsilon_\oplus(\theta) \cos\theta \sin\theta d\theta \end{aligned}$$

<u>Extended Source</u>

Now a calculation is made which takes into account the extended source, and its associated illumination function. The approximation is to assume that each element of earth surface area $dA_\oplus$ is a point source, and sum over the contributions of all illuminating elements of earth surface. We continue to assume $\epsilon_\oplus(\lambda) \simeq$ constant over the waveband of interest. Refer to Figure 4 for the geometry and variables.

The power emitted toward the sphere by surface element $dA_\oplus$, in waveband of width $d\lambda$ about λ is:

$$\epsilon_\oplus(\psi) f(T_\oplus, \lambda) \frac{\cos\psi}{\pi} dA_\oplus d\lambda \qquad \text{(W/sr)}$$

The irradiance at the sphere is then obtained by multiplying by the ratio of solid angle subtended by the sphere to its projected area:

$$\epsilon_{\oplus}(\psi) f(T_{\oplus}, \lambda) \frac{\cos\psi}{\pi} dA_{\oplus} d\lambda \frac{\delta\Omega}{\pi r^2} \qquad (\mathrm{W/cm^2})$$

From this irradiance, and the formula derived above for specular radiant intensity, we can write down the radiant intensity at observer angle ξ due to specular reflection from $dA_{\oplus}$:

$$\epsilon_{\oplus}(\psi) f(T_{\oplus}, \lambda) \frac{\cos\psi}{\pi} dA_{\oplus} d\lambda \frac{\delta\Omega}{\pi r^2} (1 - \epsilon[(\xi - \alpha)/2]) \frac{r^2}{4} \qquad (\mathrm{W/sr})$$

The total specular radiant intensity is then obtained by integrating over all illuminating $dA_{\oplus}$. Note this expression involves both the emissivity of earth, and the sphere emissivity.

We have the following relationships among variables:

$$dA_{\oplus} = R_{\oplus}^2 \sin\theta d\theta d\phi$$

$$\delta\Omega = \frac{\pi r^2}{4\pi l^2}$$

$$\cos\psi = \frac{D\cos\theta - R_{\oplus}}{l}$$

$$l^2 = R_{\oplus}^2 + D^2 - 2R_{\oplus} D \cos\theta$$

$$\alpha = \psi - \theta$$

The integration variables range $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \theta_m$, where the limit to the illuminating region is given by $\cos\theta_m = R_{\oplus}/D$.

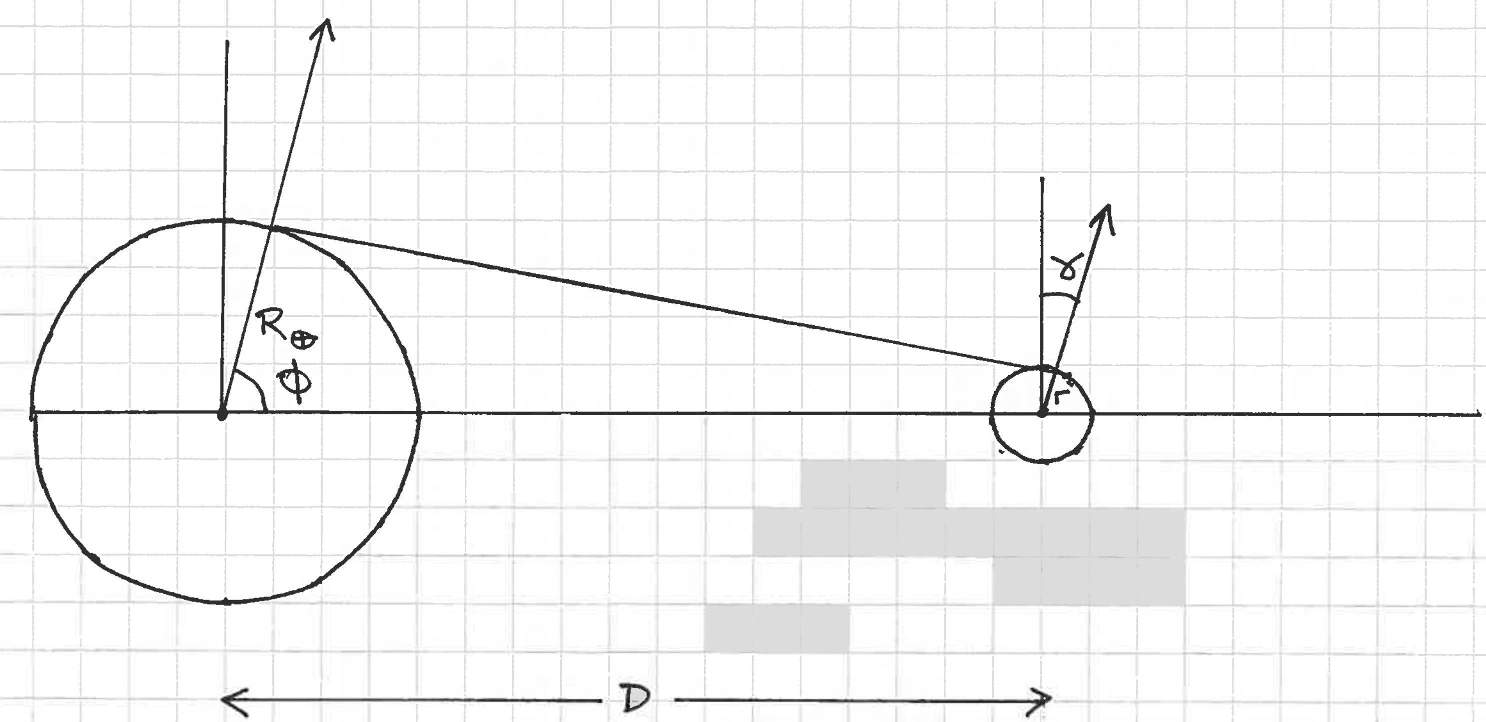
Plugging in these values to the integral, and transforming variables such that $\eta \equiv \cos\theta$, we have for the specular radiant intensity of earthshine:

$$J_{\oplus}(\xi) = \frac{R_{\oplus}^2 r^2}{8\pi} g(T_{\oplus}, \Delta\lambda) \int_{R_{\oplus}/D}^{1} d\eta \; \epsilon_{\oplus}(\eta) \frac{(D\eta - R_{\oplus})(1 - \epsilon[\xi, \eta])}{(R_{\oplus}^2 + D^2 - 2R_{\oplus} D\eta)^{3/2}} \qquad (\mathrm{W/sr})$$

Illumination Function

Figure 3

$\phi + \gamma = \pi/2$



Specular Reflection
of Extended Source

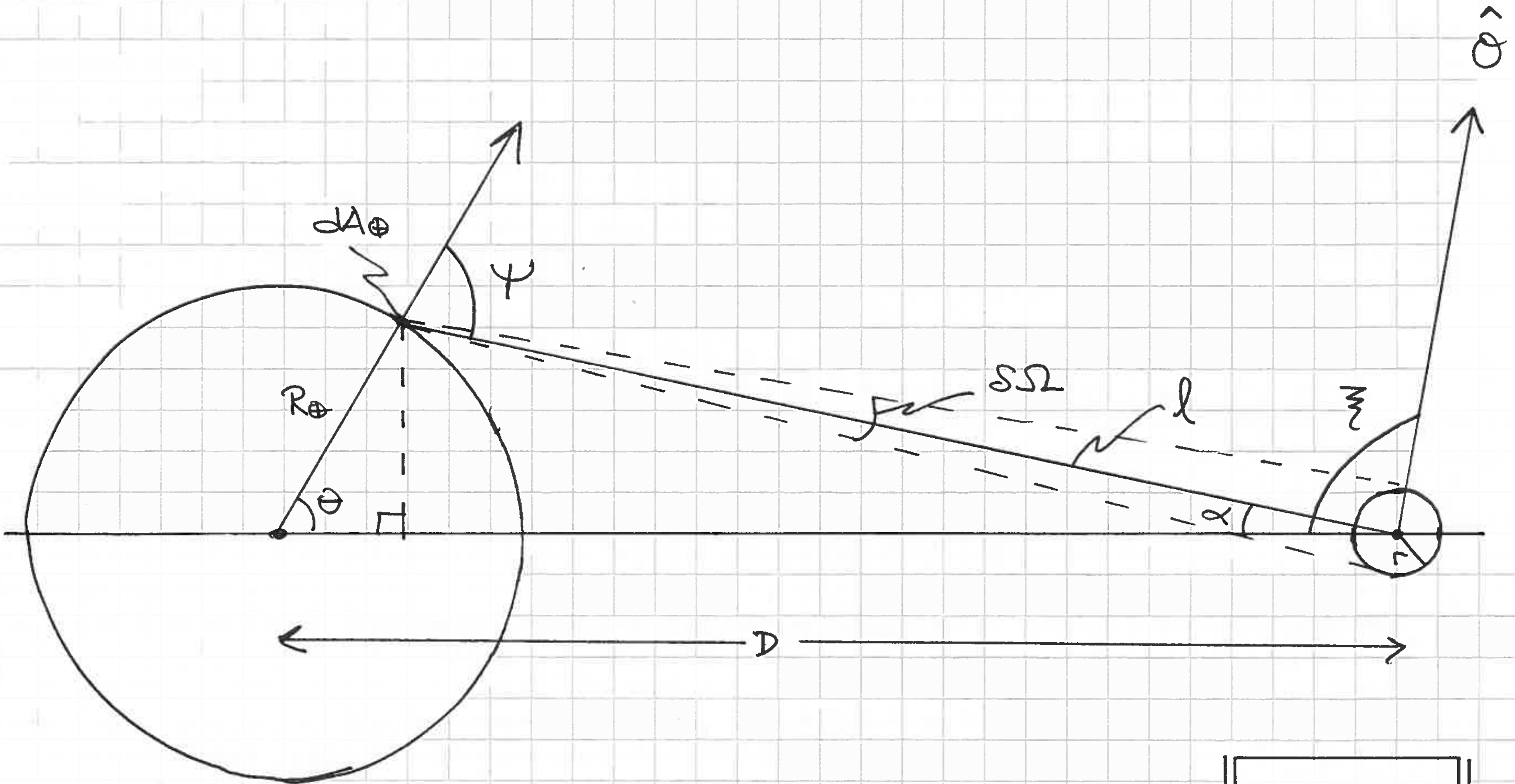


Figure 4

REFLECTED EARTH ALBEDO ESTIMATE

Here an estimate is given of the radiant intensity of sunshine reflected first off the earth, and then from the sphere to the observer. The sunshine reflected from earth will be approximated as specular, so that the intensity of radiation illuminating the sphere is:

$$F_{\odot}(1 - \bar{\epsilon}_{\oplus})\frac{R_{\oplus}^2}{4} \qquad \text{(W/sr)}$$

In reality $\epsilon_{\oplus}$ would have spatial and temporal dependence as mentioned previously, as well as the angular dependence allowed above, but for the purposes of this simple estimate we adopt a mean representative value, $\bar{\epsilon}_{\oplus}$. The specular assumption will yield an underestimate of the radiation incident on the sphere, as there will be a diffuse component to the reflected solar radiation.

As previously, the earth albedo source irradiance at the sphere is then:

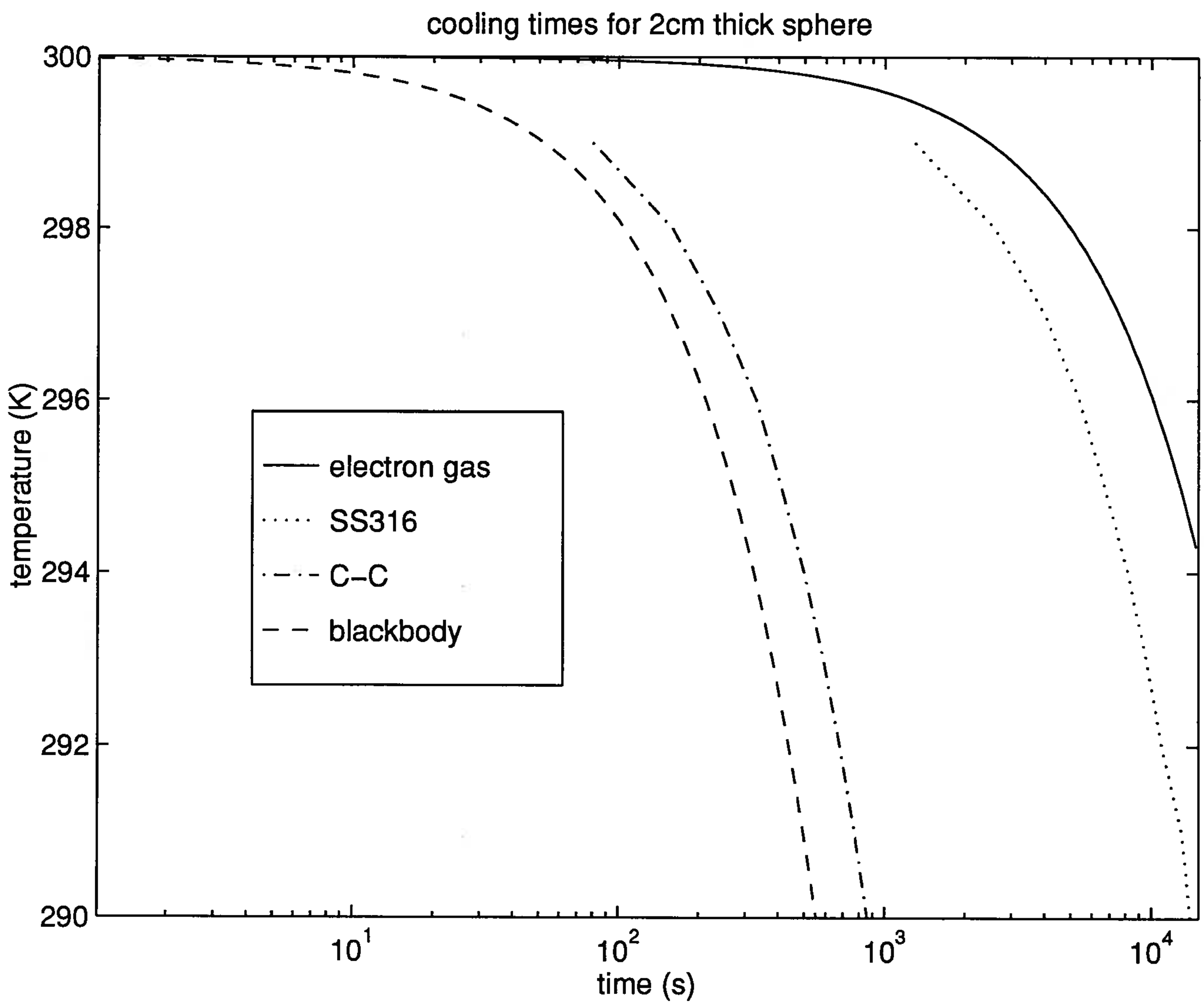
$$F_{\odot}(1 - \bar{\epsilon}_{\oplus})\frac{R_{\oplus}^2}{4}\frac{\delta\Omega}{\pi r^2} \qquad \text{(W/cm}^2\text{)}$$

where $\delta\Omega = \pi r^2/4\pi\bar{l}^2$ and $\bar{l}$ is the distance to the sphere from the reflecting earth latitude. Finally, from this irradiance, and assuming specular reflection from the sphere, the radiant intensity of reflected earth albedo is:

$$J_{\odot\oplus} \sim F_{\odot}(1 - \bar{\epsilon}_{\oplus})(1 - \bar{\epsilon})\frac{R_{\oplus}^2}{\bar{l}^2}\frac{r^2}{64\pi} \qquad \text{(W/sr)}$$

As for the earth emissivity, we have taken a mean representative value of the sphere emissivity, $\bar{\epsilon}$. The value of $\bar{l}$ will be greater than $R_{\oplus}$, perhaps much greater, depending on the sphere position. Compared to the specular reflected sunshine calculated previously, we see that the albedo term can be several orders of magnitude smaller, for values of $\bar{l} \sim 2R_{\oplus}$.

Figure 1



cooling times for 2cm thick sphere
300
298
296
294
292
290
temperature (K)
electron gas
SS316
C–C
blackbody
10^1
10^2
10^3
10^4
time (s)

Figure 2

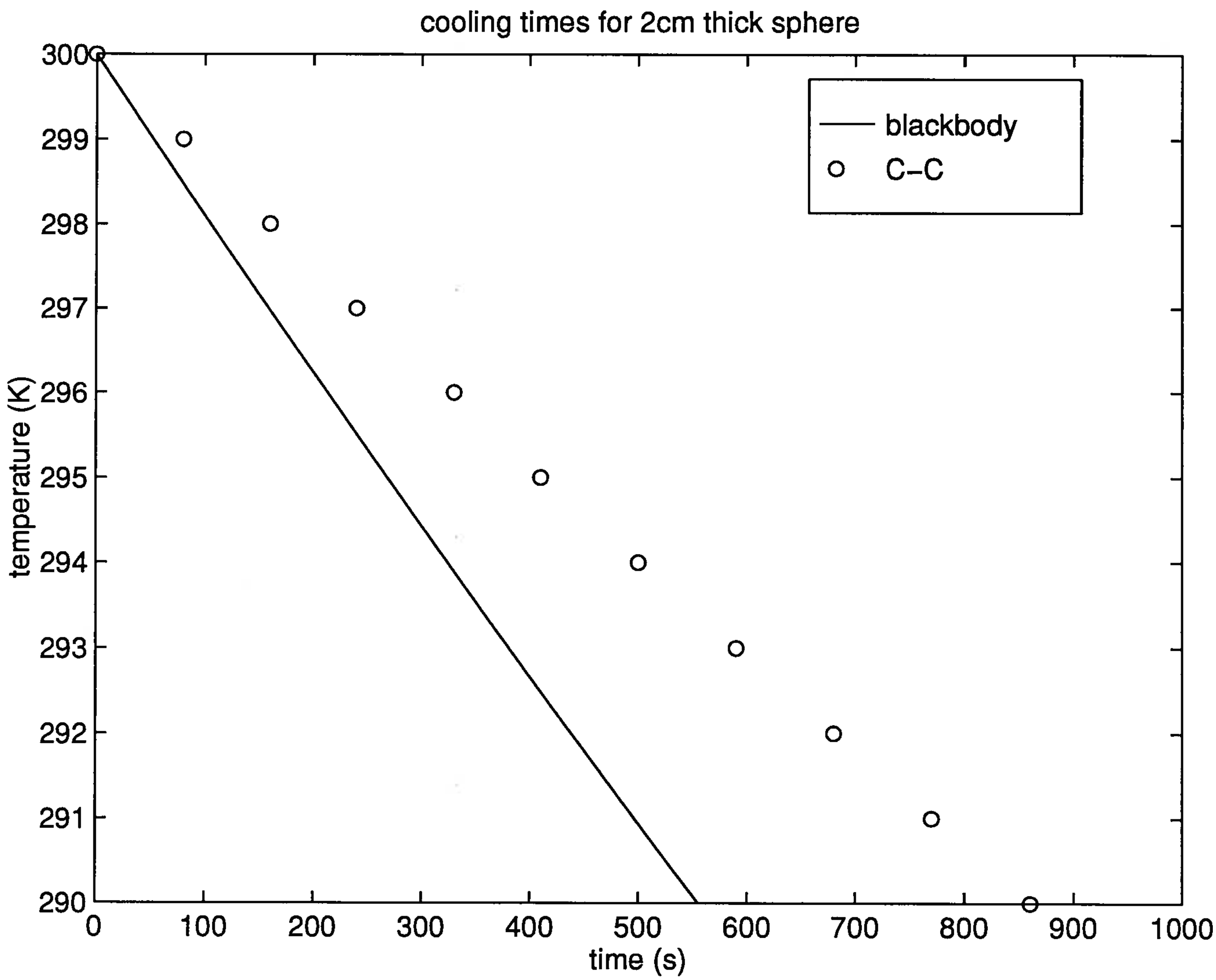
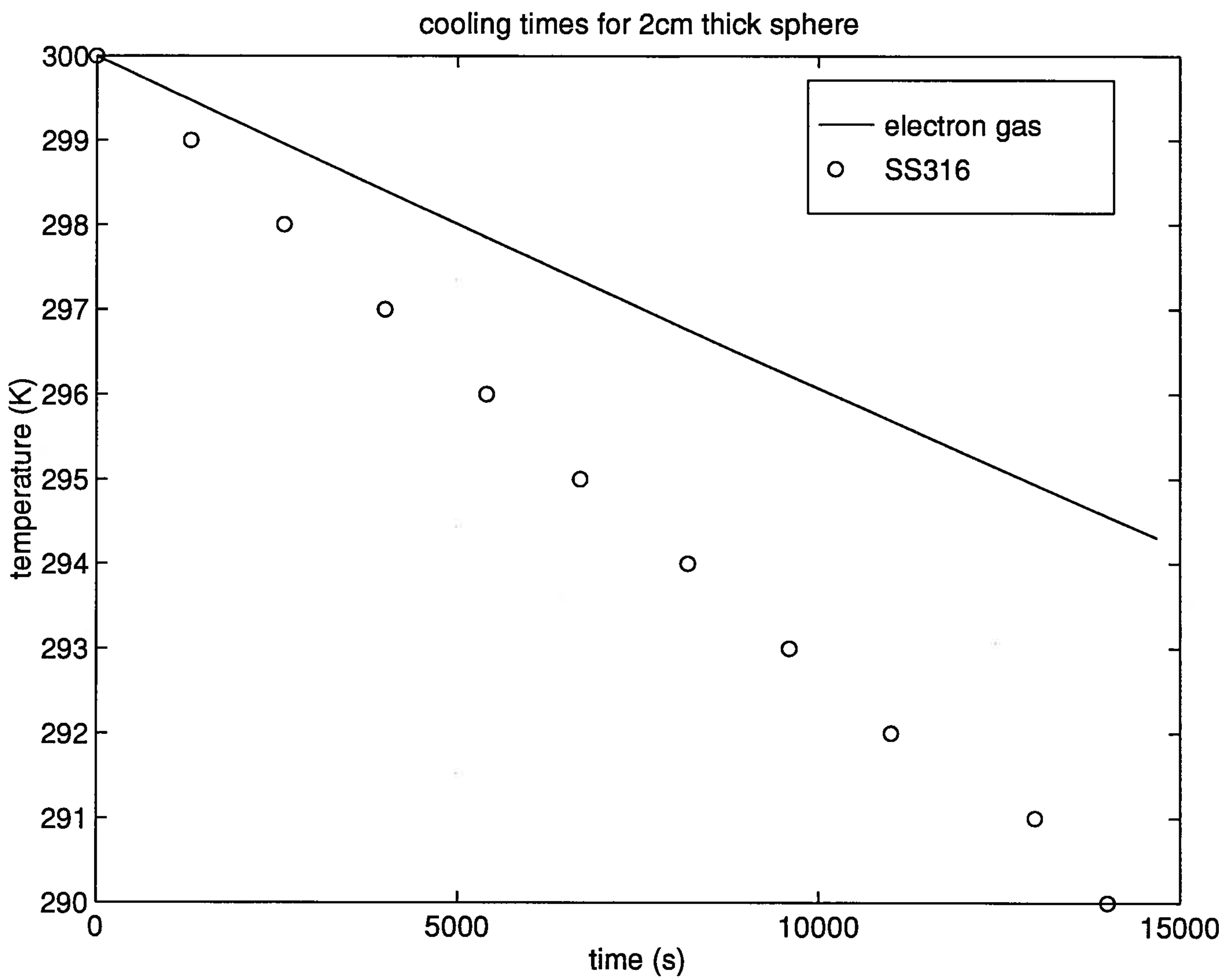
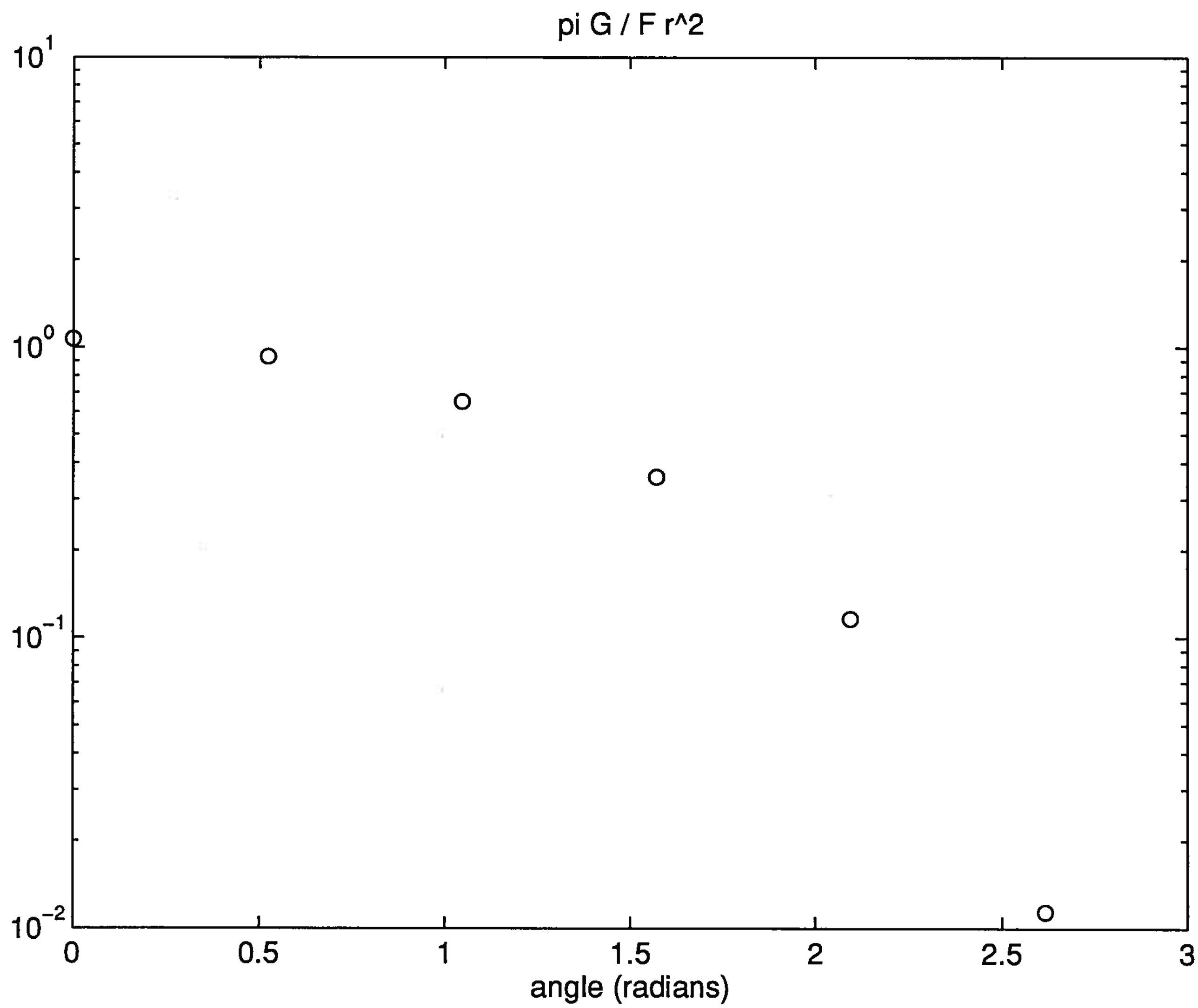

cooling times for 2cm thick sphere
300
299
298
297
296
295
294
293
292
291
290
temperature (K)
0
100
200
300
400
500
600
700
800
900
1000
time (s)
blackbody
C–C

Figure 3


cooling times for 2cm thick sphere
temperature (K)
time (s)
electron gas
SS316
300
299
298
297
296
295
294
293
292
291
290
0
5000
10000
15000



pi G / F r^2
10^1
10^0
10^-1
10^-2
0
0.5
1
1.5
2
2.5
3
angle (radians)