Physical Optics Theory of Radar Cross Section

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Physical optics provides an analytic tool for the calculation of radar cross section (RCS) for a variety of targets and circumstances. It is an improvement over the simple geometric optics treatment of ray tracing, and provides many standard expressions for the RCS of simple shapes: flat plates, cones, cylinders. It combines an expression for electromagnetic fields in terms of surface integrals, with a simplifying assumption for the surface fields. The fields, and therefore RCS, at any point in space are expressed in terms of an integral over the surface of the reflecting target. The mathematical development is also of interest because the surface integrals without any assumptions about the surface fields form the basis of the method-of-moments numerical schemes used by low-frequency RCS modelling codes (eg, FISC).

Start with the well-chosen identity for vectors $\mathbf{P}$ and $\mathbf{Q}$:

$$\nabla \cdot (\mathbf{P} \times \nabla \times \mathbf{Q}) = (\nabla \times \mathbf{P}) \cdot (\nabla \times \mathbf{Q}) - \mathbf{P} \cdot (\nabla \times \nabla \times \mathbf{Q})$$

Combine this with the expression interchanging $\mathbf{P}$ and $\mathbf{Q}$, integrate over volume, and use the divergence theorem to obtain:

$$\oint (\mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}) \cdot d\mathbf{a} = \int [\mathbf{Q} \cdot (\nabla \times \nabla \times \mathbf{P}) - \mathbf{P} \cdot (\nabla \times \nabla \times \mathbf{Q})] \, d^3x$$

Here, the surface integral is the closed boundary of the volume integral.

This equality is combined with Maxwell's equations for the electromagnetic fields:

$$\nabla \cdot \mathbf{E} = \rho_f / \epsilon \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \epsilon\mu \frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J}_f$$

The f subscript indicates the free charges and currents; material properties are contained in the values of ϵ and μ, the permittivity and permeability of the medium. Assume harmonic time dependence in the fields $\propto e^{i\omega t}$, and assume stationary and homogeneous media. Then rewrite Maxwell's equations, dropping the f subscripts:

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu\mathbf{J} - \epsilon\mu\omega^2\mathbf{E} = 0$$

$$\nabla \times \mathbf{E} + i\omega\mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - i\omega\epsilon\mu\mathbf{E} = \mu\mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{B} - \epsilon\mu\omega^2\mathbf{B} = \mu\nabla \times \mathbf{J}$$

Finally, we will need the propagator from fields and currents at position $\mathbf{x}$ to observation point $\mathbf{x}'$:

$$\phi(\mathbf{x}', \mathbf{x}) = \frac{e^{ikr}}{r}$$

where $r \equiv \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$ and $k^2 \equiv \omega^2\epsilon\mu$.

In terms of $\hat{\mathbf{r}} \equiv (\mathbf{x}' - \mathbf{x})/r$:

$$\nabla r = -\hat{\mathbf{r}} \qquad \nabla\left(\frac{1}{r}\right) = \frac{\hat{\mathbf{r}}}{r^2} \qquad \nabla\phi = \left(\frac{1}{r} - ik\right)\phi\hat{\mathbf{r}} \qquad \nabla^2\phi = -k^2\phi$$

And for an arbitrary unit vector $\hat{\mathbf{a}}$:

$$\nabla \times \phi\hat{\mathbf{a}} = \nabla\phi \times \hat{\mathbf{a}} \qquad \nabla \times \nabla \times \phi\hat{\mathbf{a}} = \nabla(\hat{\mathbf{a}} \cdot \nabla\phi) + k^2\phi\hat{\mathbf{a}}$$

Now, set $\mathbf{P} \to \mathbf{E}$ and $\mathbf{Q} \to \phi\hat{\mathbf{a}}$ in the integral identity. After substituting from Maxwell's equations, the arbitrary vector $\hat{\mathbf{a}}$ can be factored out of both expressions to yield:

$$\oint [(\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi + (\hat{\mathbf{n}} \cdot \mathbf{E})\nabla\phi - i\omega\phi(\hat{\mathbf{n}} \times \mathbf{B})]da = \int \left[\frac{\rho}{\epsilon}\nabla\phi - i\omega\mu\phi\mathbf{J}\right] d^3x$$

where $d\mathbf{a} \equiv \hat{\mathbf{n}}da$ defines the normal $\hat{\mathbf{n}}$ to the surface enclosing the volume.

The observation point $\mathbf{x}'$ is interior to the surface, and $\hat{\mathbf{n}}$ points outward from the volume (see illustration). This expression relates the surface integral of the fields to the volume integral of the sources. It is still incomplete because the singular point $r = 0$ is included in the volume. Since ϕ is not defined there, the observation point $\mathbf{x}'$ must be enclosed in a small surface, so that the volume is now contained between the large outer surface considered previously, and a small inner surface excluding the singularity. On this small surface, as for the boundary surface, the normal points out of the volume, and therefore points toward $\mathbf{x}'$, parallel to $\hat{\mathbf{r}}$. The surface integral is then decomposed into two pieces: one for the outer enclosing surface, one excluding the singularity. Let this latter surface be spherical, of radius r_2. Then $da = 4\pi r_2^2$. Evaluate all terms and take the limit $r_2 \to 0$. Since $(\hat{\mathbf{n}} \times \mathbf{E}) \times \hat{\mathbf{n}} + (\hat{\mathbf{n}} \cdot \mathbf{E})\hat{\mathbf{n}} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{E} = \mathbf{E}$, it is found that the singular integral reduces to $4\pi\mathbf{E}(\mathbf{x}')$, so that:

$$\boxed{\mathbf{E}(\mathbf{x}') = \frac{1}{4\pi}\int \left[\frac{\rho}{\epsilon}\nabla\phi - i\omega\mu\phi\mathbf{J}\right] d^3x - \frac{1}{4\pi}\oint [(\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi + (\hat{\mathbf{n}} \cdot \mathbf{E})\nabla\phi - i\omega\phi(\hat{\mathbf{n}} \times \mathbf{B})]da}$$

Going through the same process with $\mathbf{P} \to \mathbf{B}$ yields:

$$\boxed{\mathbf{B}(\mathbf{x}') = \frac{1}{4\pi}\int \mu\mathbf{J} \times \nabla\phi \; d^3x - \frac{1}{4\pi}\oint [(\hat{\mathbf{n}} \times \mathbf{B}) \times \nabla\phi + (\hat{\mathbf{n}} \cdot \mathbf{B})\nabla\phi + i\omega\epsilon\mu\phi(\hat{\mathbf{n}} \times \mathbf{E})]da}$$

These equations constitute an alternative expression of Maxwell's equations, assuming only harmonic time variations, and homogeneous stationary media. They generally describe the fields in space as due to contributions from sources distributed through all space, and the values of fields on an enclosing boundary.

In some situations, fields may be sought which are produced by sources alone, and the boundary extends to infinity. It can then be assumed that surface contributions vanish at infinity, and the fields due to sources alone are:

$$\mathbf{E}_{src}(\mathbf{x}') = \frac{1}{4\pi}\int \left[\frac{\rho}{\epsilon}\nabla\phi - i\omega\mu\phi\mathbf{J}\right] d^3x \qquad \mathbf{B}_{src}(\mathbf{x}') = \frac{1}{4\pi}\int \mu\mathbf{J}\times\nabla\phi \; d^3x$$

In other situations, no sources exists, and the fields are sought in terms of boundary fields. In this case, the fields at the observation point are given in terms of fields on the boundary:

$$\mathbf{E}_{bnd}(\mathbf{x}') = \frac{1}{4\pi}\oint [i\omega\phi(\hat{\mathbf{n}}\times\mathbf{B}) - (\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\phi - (\hat{\mathbf{n}}\cdot\mathbf{E})\nabla\phi]da$$

$$\mathbf{B}_{bnd}(\mathbf{x}') = -\frac{1}{4\pi}\oint [(\hat{\mathbf{n}}\times\mathbf{B})\times\nabla\phi + (\hat{\mathbf{n}}\cdot\mathbf{B})\nabla\phi + i\omega\epsilon\mu\phi(\hat{\mathbf{n}}\times\mathbf{E})]da$$

This is the case of interest for calculation of radar cross section.

Although the formalism was developed for the boundary external to $\mathbf{x}'$, the results are valid for a world turned inside out, with $\mathbf{x}'$ outside the boundary. Then the boundary surface encloses the part of space without sources, and the volume is the rest of all space. This geometry is particularly suited to calculating the radiation reflected by some target body. The boundary then coincides with the surface of the body reflecting radiation to the observer. The boundary normal points inward on the body, still outward from the 'enclosing' volume.

These surface integrals form the basis of numerical method-of-moments schemes for RCS calculation. Such models combine information about surface properties, perhaps an impedance condition, to produce the surface fields from the known incident radiation field. The surface is then numerically decomposed and the fields at the observation point are produced as a sum over the surface integral elements.

However, these results apply only for a closed boundary surface. One interesting problem would be the fields produced by an opening in an opaque screen, a diffraction slit for example. Such a source, considered as a surface, would not be closed. Now, one could apply the formalism to an open surface if it could be considered part of a larger closed surface, with only the open part having a non-zero contribution. But we also implicitly assume the fields and their first derivatives are continuous over the surface. To break the surface into an open region bounded by a closing surface which makes zero contribution to the fields, one may find that discontinuous jumps in the fields across the boundary separating the two regions are required. Discontinuities in the field will imply discontinuities in the surface currents. This discontinuity in the surface current across the boundary enclosing the open contribution will in turn imply a line charge density along the boundary.

By comparing the surface and volume integrals, and considering a limit in which the volume integral of the volume charge and current densities is expressed as a surface integral of surface current density $\mathbf{K}$ and surface charge density τ, we can identify in the surface terms an effective $\mathbf{K} = -\hat{\mathbf{n}} \times \mathbf{B}/\mu$, and an effective $\tau = -\hat{\mathbf{n}} \cdot \mathbf{E}\epsilon$. In this way, part of the boundary contribution can be interpreted as due to surface sources. In fact, the general jump conditions for electric and magnetic fields across a surface separating media 1 and 2, with the normal pointing toward medium 2, are:

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \qquad \hat{\mathbf{n}} \times (\mathbf{B}_2/\mu_2 - \mathbf{B}_1/\mu_1) = \mathbf{K}$$

$$\hat{\mathbf{n}} \cdot (\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1) = \tau \qquad \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

Comparing these relations with those inferred from the integrals implies the fields vanish outside the boundary.

Now, it is these surface 'sources' that would be discontinous across the boundary of an open surface. The equation of charge conservation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

will hold across the boundary, and will imply in 2 dimensions:

$$-i\omega\tau = \hat{\mathbf{n}}_1 \cdot (\mathbf{K}_1 - \mathbf{K}_2) = -\hat{\mathbf{n}}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{B}_1)/\mu_1 = -\mathbf{B}_1 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}})/\mu_1$$

where $\hat{\mathbf{n}}_1$ is a unit vector in the plane of the boundary surface and normal to the boundary contour enclosing the open source region of the surface. The value for the surface current has been substituted in from above, and the field outside the open source region has been set to zero.

Thus the fields produced by an open surface region will include a source of line charge density bounding the open surface

$$\tau = \mathbf{B}_1 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}})/i\mu_1\omega$$

We want to generalize the source volume integral of charge density to allow for this line charge density:

$$\frac{1}{4\pi}\int \frac{\rho}{\epsilon}\nabla\phi d^3x \Longrightarrow \frac{1}{4\pi}\oint \frac{\tau}{\epsilon}\nabla\phi dl = \frac{1}{4\pi i\omega\mu_1\epsilon}\oint \nabla\phi \mathbf{B}_1 \cdot d\mathbf{l}$$

where we have used that $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}$ is parallel to the contour direction. So the electric field produced by non-zero fields on an open surface, in a source free region of space is:

$$\mathbf{E}_{opn}(\mathbf{x}') = \frac{1}{4\pi}\oint \frac{(\nabla\phi)\mathbf{B}\cdot d\mathbf{l}}{i\omega\mu\epsilon} + \frac{1}{4\pi}\int [i\omega\phi(\hat{\mathbf{n}} \times \mathbf{B}) - (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla\phi - (\hat{\mathbf{n}} \cdot \mathbf{E})\nabla\phi]da$$

where the closed contour bounds the open surface. This correction is necessary to satisfy Gauss' law for open surfaces, and the contour term goes to zero in the limit the surface is closed.

Stratton derives the corresponding modification of the magnetic field by using the artificial concept of magnetic charge, and proceeding in complete analogy with the electric field modification:

$$\mathbf{B}_{opn}(\mathbf{x}') = -\frac{1}{4\pi}\oint \frac{\phi\mathbf{E}\cdot d\mathbf{l}}{i\omega} - \frac{1}{4\pi}\int [(\hat{\mathbf{n}} \times \mathbf{B}) \times \nabla\phi + (\hat{\mathbf{n}} \cdot \mathbf{B})\nabla\phi + i\omega\epsilon\mu\phi(\hat{\mathbf{n}} \times \mathbf{E})]da$$

Unless a numerical scheme is implemented, the above integrals will be largely intractable. So the theory of physical optics (PO) introduces a couple of simplifying assumptions to the surface integrals. The first is the far field approximation: the size of the object is assumed to be much smaller than its distance from the observation point. This assumption is robust for the purposes of calculating RCS. Then the origin of coordinates is put in or near to the object surface. Defining the observation point position $\mathbf{R} \equiv \mathbf{x}'$, we assume $R \gg x$. Then, for $\mathbf{r} = \mathbf{R} - \mathbf{x}$, $r = R - \hat{\mathbf{R}} \cdot \mathbf{x} + O(x^2/R^2)$, so that:

$$\hat{\mathbf{r}} = \hat{\mathbf{R}} + \frac{\mathbf{R}(\hat{\mathbf{R}} \cdot \mathbf{x})}{R^2} - \frac{\mathbf{x}}{R} + O\left(\frac{x^2}{R^2}\right)$$

These results allow us to approximate the propagator and its derivative:

$$\phi \simeq \frac{\mathrm{e}^{ikR}}{R} \mathrm{e}^{-ik\hat{\mathbf{R}} \cdot \mathbf{x}} \qquad \nabla\phi \simeq -ik\phi\hat{\mathbf{R}}$$

where we have made the reasonable assumption that the distance to the target is much larger than the wavelength, $k \gg 1/r$. Also note we used the first order expressions only in the phase of ϕ, using zeroth order approximations elsewhere. This is because effects of phase variations will dominate effects of distance variations in the calculated field.

The second approximation underlying PO is more stringent: the tangent plane approximation. The surface fields, required to evaluate the field integrals, are approximated by the values that would obtain if each surface patch were infinite and flat, allowing one to use the Fresnel equations to express the surface fields in terms of the incident fields, the surface electrical properties, and the incident angle relative to the surface patch normal. Where the surface is shadowed from the incident radiation, the surface fields are set to zero. The tangent plane approximation will require that the 'radius of curvature' of all surface elements are much larger than the incident radiation wavelength, so that the flat plane assumption is valid. This is a poor assumption at UHF frequencies, but not too bad at Xband. Thus the surface integrals are based on surface fields obtained from a geometric optics treatment.

Even so, PO does produce realistic returns as a function of aspect angle, and provides a firm basis for interpreting the results of numerical method-of-moments schemes. The PO functional forms can provide a basis function or functions for ordering the numerical results. This function fitting for real objects is taken up in another report.

Here are the Fresnel expressions for the surface fields (assuming infinite planar surfaces). The surface is in the $\hat{\mathbf{y}} - \hat{\mathbf{z}}$ plane, and the $\hat{\mathbf{x}}$ direction points into the target. Radiation is incident at angle θ_i with respect to the surface normal; θ_t is the angle of the transmitted wave (which may in general be a complex number). The index of refraction of the target medium is n_2, the incident medium n_1 (typically air or vacuum).

For the electric vector perpendicular to the incidence plane:

$$\frac{E_s}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{B_{sx}}{B_{ix}}$$

$$\frac{B_{sy}}{B_{iy}} = \frac{2n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

For the electric vector in the incidence plane:

$$\frac{n_2}{n_1}\frac{E_{sx}}{E_{ix}} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{E_{sy}}{E_{iy}}$$

$$\frac{B_s}{B_i} = \frac{2n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

These expressions, too, are a bit cumbersome. Since we have exposed ourselves to the inaccuracy of Fresnel surface fields, it will not do much more harm to our approximation to use the Fresnel results for a perfect electrical conductor (PEC), $n_2 \gg n_1$. Although targets of interest may be coated with dielectrics of various sorts (eg, heat shields), the underlying structures are metallic. Unless the target is without underlying metal structure, or is tuned with a radar absorbing material at the frequency of interest, a PEC approximation will give valid results within the context of the tangent plane approximation to the surface fields. The Fresnel equations for the PEC surface fields are simply:

$$\hat{\mathbf{n}} \times \mathbf{E}_s = 0 \qquad \hat{\mathbf{n}} \times \mathbf{B}_s = 2\hat{\mathbf{n}} \times \mathbf{B}_i$$

$$\hat{\mathbf{n}} \cdot \mathbf{E}_s = 0 \qquad \hat{\mathbf{n}} \cdot \mathbf{B}_s = 0$$

The electric field and the normal component of the magnetic field vanish at the surface. The tangential magnetic field is just twice the incident magnetic field.

Putting these approximations into the expressions for the closed surface integrals yields:

$$\boxed{\mathbf{E}_{PO}(\mathbf{R}) = \frac{i\omega}{2\pi}\frac{\mathrm{e}^{ikR}}{R}\oint(\hat{\mathbf{n}} \times \mathbf{B}_i)\mathrm{e}^{-ik\hat{\mathbf{R}}\cdot\mathbf{x}}da}$$

$$\boxed{\mathbf{B}_{PO}(\mathbf{R}) = \frac{ik}{2\pi}\frac{\mathrm{e}^{ikR}}{R}\oint(\hat{\mathbf{n}} \times \mathbf{B}_i) \times \hat{\mathbf{R}}\ \mathrm{e}^{-ik\hat{\mathbf{R}}\cdot\mathbf{x}}da = \frac{1}{c}\mathbf{E}(\mathbf{R}) \times \hat{\mathbf{R}}}$$

These expressions are straightforward to evaluate for simple shapes, and we will do so below. Recall that $\mathbf{E}_i = c\mathbf{B}_i \times \hat{\mathbf{k}}$.

The practical use of these approximations to the fields is to calculate RCS. In general, the RCS will depend on both field polarizations. In practice, a weakness of PO is that it allows for no cross-polarization RCS terms. In terms of a field component $\mathbf{F}$ that can be either the electric or magnetic field, the RCS is defined:

$$\sigma \equiv 4\pi R^2 \frac{|\mathbf{F}(\mathbf{R})|^2}{|\mathbf{F}_{\mathrm{i}}|^2}$$

CASE 1: Rectangular flat plate: monostatic: principal plane

Consider the reflecting face in the $\hat{\mathbf{y}} - \hat{\mathbf{z}}$ plane at $x = 0$. The observation point is at $z = 0$ at an angle θ with respect to the $\hat{\mathbf{x}}$ axis (see illustration). When $\hat{\mathbf{R}} \times \hat{\mathbf{n}}$ is parallel to a plate edge, the observation point is said to be in a *principal plane.* The initial polarization is chosen such that $\mathbf{E}_i$ is along $\hat{\mathbf{z}}$. Expressing the various quantities in these coordinates:

$$\hat{\mathbf{R}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} = -\hat{\mathbf{k}} \text{ (monostatic)}$$

$$\hat{\mathbf{n}} = -\hat{\mathbf{x}}$$

$$\mathbf{B}_i = B_i e^{i\mathbf{k}\cdot\mathbf{x}}(\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{x}}) \equiv B_i\hat{\mathbf{B}}_i e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$(\hat{\mathbf{n}} \times \mathbf{B}_i) \times \hat{\mathbf{R}} = -\cos\theta B_i\hat{\mathbf{B}}_i e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\mathbf{x} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The PO integral reduces to:

$$\begin{aligned}\mathbf{B}_{PO}(\mathbf{R}) &= \frac{-ik}{2\pi}\frac{e^{ikR}}{R}\cos\theta B_i\hat{\mathbf{B}}_i\int_{-a/2}^{a/2}dz\int_{l/2}^{l/2}e^{-2iky\sin\theta}dy \\ &= \frac{-ie^{ikR}}{\lambda R}\cos\theta a l B_i\frac{\sin(kl\sin\theta)}{kl\sin\theta}\hat{\mathbf{B}}_i\end{aligned}$$

In terms of plate area $A \equiv al$:

$$\boxed{\sigma = 4\pi\frac{A^2}{\lambda^2}\left|\cos\theta\frac{\sin(lk\sin\theta)}{lk\sin\theta}\right|^2}$$

This is the monostatic RCS in a principal plane of a flat rectangular PEC plate. The associated electric field at $\mathbf{R}$ would be:

$$\mathbf{E}_{PO}(\mathbf{R}) = -i\nu\frac{e^{ikR}}{R}\cos\theta a l B_i\frac{\sin(kl\sin\theta)}{kl\sin\theta}\hat{\mathbf{z}}$$

This electric field will produce the same RCS value.

CASE 2: Rectangular flat plate: bistatic: normal incidence

In this case, the incident radiation is normal to the plate. The angle θ is the observation angle with respect to the surface normal. With the same coordinate geometry as Case 1:

$$\hat{\mathbf{k}} = -\hat{\mathbf{x}}$$

$$\mathbf{B}_i = \hat{\mathbf{y}}$$

$$\hat{\mathbf{n}} \times \hat{\mathbf{B}}_i = -\hat{\mathbf{z}}$$

The PO integral yields for this case:

$$\mathbf{E}_{PO}(\mathbf{R}) = -i\nu\frac{e^{ikR}}{R}AB_i\frac{\sin(k(l/2)\sin\theta)}{k(l/2)\sin\theta}\hat{\mathbf{z}}$$

The associated RCS:

$$\boxed{\sigma = 4\pi\frac{A^2}{\lambda^2}\left|\frac{\sin(k(l/2)\sin\theta)}{k(l/2)\sin\theta}\right|^2}$$

This is the bistatic RCS of a rectangular PEC plate with normally-incident radiation. It is quite similar to the monostatic expression of Case 1. The two results are of course identical at $\theta = 0$.

Note that for $\theta \ll 1$, $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. In this case, the bistatic RCS at θ is equal to the monostatic RCS at $\theta/2$. This is an example of the theorem attributed to Crispin & Maffett that for PO current distributions, the bistatic RCS at angle β is approximately the monostatic RCS at $\beta/2$. This result underlies the bistatic approximation implemented in the FISC radar model, where monostatic RCS values are approximated by bistatic

RCS values for a relatively small number of current distributions, instead of calculating different current distributions for each monostatic angle. Crispin & Maffett claim their theorem holds for values of $\beta \leq 130$ degrees.

CASE 3: Circular flat plate: monostatic

Consider now a circular plate of diameter a. The setup is the same as for Case 1, except the area integral will be different:

$$\mathbf{E}_{PO}(\mathbf{R}) = \frac{-i\omega}{2\pi}\frac{e^{ikR}}{R}\cos\theta B_i\hat{\mathbf{z}}\int_{-a/2}^{a/2} e^{-2iky\sin\theta}dy\int_{-\sqrt{a^2/4-y^2}}^{\sqrt{a^2/4-y^2}} dz$$

$$= -i\omega\left(\frac{a}{2}\right)^2\frac{e^{ikR}}{R}\cos\theta B_i\frac{J_1(ka\sin\theta)}{ka\sin\theta}\hat{\mathbf{z}}$$

where J_1 is a Bessel function. The corresponding RCS:

$$\boxed{\sigma = 16\pi\frac{A^2}{\lambda^2}\left|\cos\theta\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right|^2}$$

CASE 4: Sphere: monostatic

Consider now a sphere of radius a centered at the origin, with:

$$\hat{\mathbf{B}}_i = \hat{\mathbf{y}}$$

$$\hat{\mathbf{R}} = \hat{\mathbf{z}} = -\hat{\mathbf{k}}$$

$$\mathbf{r} = a\hat{\mathbf{r}} = a(\cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$$

$$\hat{\mathbf{n}} = -\hat{\mathbf{r}}$$

$$\mathbf{B}_i = B_i\hat{\mathbf{B}}_i e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\hat{\mathbf{n}} \times \hat{\mathbf{B}}_i = \cos\theta\hat{\mathbf{x}} - \cos\phi\sin\theta\hat{\mathbf{z}}$$

where θ and ϕ are the standard spherical polar angle coordinates. The fields are taken to be non-zero only on the illuminated side $\theta \leq \pi/2$. Then:

$$\begin{aligned}\mathbf{E}_{PO}(\mathbf{R}) &= i\nu\frac{\mathrm{e}^{ikR}}{R}B_i a^2 \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi(\cos\theta\hat{\mathbf{x}} - \cos\phi\sin\theta\hat{\mathbf{z}})\mathrm{e}^{-2iak\cos\theta} \\ &= i\nu\frac{\mathrm{e}^{ikR}}{R}B_i 2\pi a^2 \left[\frac{1}{\xi^2} - \frac{\mathrm{e}^{\xi}}{\xi}\left(\frac{1}{\xi} - 1\right)\right]\hat{\mathbf{x}}\end{aligned}$$

where $\xi \equiv -2iak$.

The second term in brackets corresponds to the integral evaluated at $\theta = 0$, and represents the specular flash from the front of the sphere. Since PO requires $ak \gg 1$, the term of order $1/\xi$ will dominate the expression. An approximation to the RCS is therefore:

$$\boxed{\sigma = \pi a^2 + O(1/\xi^3)}$$

demonstrating agreement with the optical limit expected from the exact Mie series expression.

The other term of $O(1/\xi^2)$ originated from the integral at $\theta = \pi/2$. This term is in fact spurious, and arises from the shadow boundary. This illustrates a fundamental limitation of PO: the discontinuous illumination function implies spurious currents that will contribute to the integrals. Their nature is exactly as considered above for open surfaces. A line integral around the open illuminated surface will not correct this because the fields really *do not* drop to zero in the shadowed region, as they would for an opaque screen. And this brings up another limitation: creeping waves, which we know exist on the sphere from the Mie series expression, are not contained in a treatment which sets the fields in the shadowed region to zero. So PO recklessly applied can ignore real contributions to RCS (creeping waves), while including other spurious contributions from the shadow boundary.

PO expressions will typically incorporate one more approximation which has the advantage of both simplifying the integrals and at the same time ignoring the phase variations which give rise to spurious shadow currents. This is the stationary phase approximation (SP):

$$\int_a^b g(\phi)\mathrm{e}^{ikh(\phi)}d\phi \simeq \mathrm{e}^{ikh_0}\frac{g(\phi_0)}{k^{1/2}}\sqrt{\frac{2\pi}{h_0''}}\mathrm{e}^{i\pi/4}$$

which is valid in the limit $k \to \infty$ (see Appendix). In this limit, the exponential term oscillates rapidly, and the integral averages to zero over much of its range. The main contribution will come from stationary points ϕ_0 where $dh(\phi_0)/d\phi = 0$. Rapidly oscillating terms near the shadow boundary, and the associated spurious contributions, are ignored. So, let's revisit the RCS of a sphere using SP.

CASE 5: Sphere: monostatic: stationary phase

The vagaries of these integral expressions, exact or approximate, are such that tractability depends on coordinate choices. Although the sphere is aspect-independent, spherical polar coordinates do have a preferred direction and a singular point. Therefore, to get stationary phase to work, we must adopt different coordinates than we used for the exact expression. (It turns out these new coordinates would yield intractable integrals for the exact expression!) Now, choose:

$$\hat{\mathbf{B}}_i = \hat{\mathbf{z}}$$

$$\hat{\mathbf{R}} = \hat{\mathbf{x}} = -\hat{\mathbf{k}}$$

Then:

$$\begin{aligned}
\mathbf{E}_{PO}(\mathbf{R}) &= i\nu\frac{\mathrm{e}^{ikR}}{R}B_i a^2\int_0^\pi \sin\theta d\theta\int_{-\pi/2}^{\pi/2} d\phi(\cos\phi\sin\theta\hat{\mathbf{y}} - \sin\phi\sin\theta\hat{\mathbf{x}})\mathrm{e}^{-2iak\cos\phi\sin\theta} \\
&\simeq i\nu\frac{\mathrm{e}^{ikR}}{R}B_i a^2\mathrm{e}^{i\pi/4}\hat{\mathbf{y}}\int_0^\pi \sin^2\theta\sqrt{\frac{2\pi}{2ak\sin\theta}}\mathrm{e}^{-2iak\sin\theta}d\theta \qquad (\phi_0 = 0) \\
&\simeq \nu\frac{\mathrm{e}^{ikR}}{R}B_i\mathrm{e}^{-2iak}\frac{a\pi}{k}\hat{\mathbf{y}} \qquad (\theta_0 = \pi/2)
\end{aligned}$$

Here, the stationary point is two-dimensional, occurring as indicated above. The stationary point corresponds to just the specular point. And,

$$\sigma = \pi a^2$$

the optical limit, and the leading term of the exact calculation.

Thus, SP simplified the integrals and ignored spurious shadow boundary contributions. Of course, the creeping wave effects are still missing. We shall incorporate the SP approximation into our evaluation of PO integrals for curved surfaces. We cannot use the approximation for flat surfaces because the first derivative of the exponent does not vanish on the surface. Fortunately, the vanishing of the fields at the edge is a valid assumption, and no spurious currents are introduced. The SP requirement that the argument of the exponential $\gg 1$ is consistent with the tangent plane approximation. One however must be careful that angular arguments to the exponential do not vanish at the stationary point, or else the SP assumption will be violated. (That's why we had to pick different coordinates than for the exact calculation).

CASE 6: Cylinder: monostatic: stationary phase

Since this surface is curved, we shall avoid spurious contributions and go right to a SP approximation. Consider the reflection from the surface of a cylinder of length l and radius a, at angle θ with respect to the surface normal (see illustration). We ignore reflection from the inside of the surface that may be seen at oblique angles. Orient the cylinder axis along $\hat{\mathbf{z}}$ and work in cylindrical coordinates:

$$\hat{\mathbf{R}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}} = -\hat{\mathbf{k}}$$

$$\mathbf{x} = a\cos\phi\hat{\mathbf{x}} + a\sin\phi\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\hat{\mathbf{n}} = -\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}}$$

$$\mathbf{B}_i = B_i e^{i\mathbf{k}\cdot\mathbf{x}}(\cos\theta\hat{\mathbf{z}} - \sin\theta\hat{\mathbf{x}})$$

so that, integrating over the illuminated region:

$$\mathbf{E}_{PO}(\mathbf{R}) = i\nu\frac{e^{ikR}}{R}B_i\int_{-\pi/2}^{\pi/2}ad\phi\int_{1/2}^{1/2}ldq[-\sin\phi\cos\theta\hat{\mathbf{x}} + \cos\phi\cos\theta\hat{\mathbf{y}} - \sin\phi\sin\theta\hat{\mathbf{z}}]* \\ * e^{-2ik(a\cos\phi\cos\theta + z\sin\theta)}$$

As with the sphere, we have a 2D phase variation. But since there are two terms in the phase, we must be sure each is asymptotically large to employ SP. In fact, for the entire range of θ, both terms cannot be large simultaneously. For near-normal incidence, we can have $ak\cos\theta \to \infty$; for near-tangential incidence, we can have $klq\sin\theta \to \infty$. Only for intermediate incidence angles can both terms be large (assuming ka and kl are large). But it turns out that no stationary point exists for $klq\sin\theta$, since the derivative does not vanish with respect to q. So if we use SP to get a result free from spurious fields, the result will not be valid near $\theta = \pi/2$. Therefore we require $ak\cos\theta \to \infty$. The stationary point occurs at $\phi = 0 \equiv \phi_0$:

$$\begin{aligned}\mathbf{E}_{PO}(\mathbf{R}) &\simeq i\nu\frac{e^{ikR}}{R}B_i al\cos\theta e^{-2iak\cos\theta}\sqrt{\frac{2\pi}{2ak\cos\theta}}e^{i\pi/4}\hat{\mathbf{y}}\int_{-1/2}^{1/2}e^{-2iklq\sin\theta}dq \\ &= i\nu\frac{e^{ikR}}{R}B_i e^{i\pi/4}e^{-2iak\cos\theta}l\sqrt{\frac{2\pi}{2ak\cos\theta}}\frac{\sin(lk\sin\theta)}{lk\sin\theta}\hat{\mathbf{y}}\end{aligned}$$

Where the final integral was done exactly. The corresponding RCS:

$$\boxed{\sigma = kal^2\cos\theta\left|\frac{\sin(lk\sin\theta)}{lk\sin\theta}\right|^2}$$

Except for the $\cos\theta$ term, this is the standard PO result for the monostatic RCS of a cylinder. Because the $\sin x/x$ term goes to zero near $\theta = \pi/2$, the $\cos\theta$ factor can be ignored at less than 1% error.

CASE 7: Cone: monostatic: stationary phase

Consider reflection from a cone of height h and half-angle α, centered at the origin with symmetry axis along $\hat{\mathbf{z}}$, viewed at monostatic angle θ with respect to the $\hat{\mathbf{z}}$ axis (see illustration). We ignore contributions from the inside of the cone that may be seen at basal aspects, or any assumed basal surface (the reflection from a circular disk was considered previously). In cylindrical coordinates:

$$\mathbf{x} = r(z)\cos\phi\hat{\mathbf{x}} + r(z)\sin\phi\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$r(z) = (h/2 - z)\tan\alpha$$

$$\hat{\mathbf{R}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} = -\hat{\mathbf{k}}$$

$$\mathbf{B}_i = \cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$-\hat{\mathbf{n}} = \cos\phi\cos\alpha\hat{\mathbf{x}} + \sin\phi\cos\alpha\hat{\mathbf{y}} + \sin\alpha\hat{\mathbf{z}}$$

An infinitesimal element of surface area $dA = dz\sqrt{1+\tan^2\alpha}\,r(z)d\phi$, and the electric field integral:

$$\mathbf{E}_{PO}(\mathbf{R}) = i\nu\frac{e^{ikR}}{R}B_i\int_{-h/2}^{h/2}dz\int_{\phi_1}^{\phi_2}d\phi\left(\frac{h}{2}-z\right)\frac{\tan\alpha}{\cos\alpha}[(\cos\theta\sin\alpha+\sin\theta\cos\alpha\sin\phi)\hat{\mathbf{x}}$$
$$-\sin\theta\cos\alpha\cos\phi\hat{\mathbf{y}} - \cos\theta\cos\alpha\cos\phi\hat{\mathbf{z}}]e^{-2ik[(h/2-z)\sin\alpha\sin\phi\sin\theta/\cos\alpha+z\cos\theta]}$$

The integration limits on ϕ are determined by the illumination function, which depends on θ. The shadow boundary exists where $\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = 0 = \sin\theta\sin\phi_b\cos\alpha + \sin\alpha\cos\theta$, which implies the boundary occurs at $\sin\phi_b = -\tan\alpha/\tan\theta$. As $\theta \to 0$, $\phi_b \to 3\pi/2$, and the shadow boundary wraps all the way around the cone. As $\theta \to \pi$, $\phi_b \to \pi/2$, and the illuminated cone surface area goes to zero. Stationary points occur at $\phi = \pi/2, 3\pi/2$. Both are illuminated at $\theta = 0$, neither at $\theta = \pi$. We then restrict consideration to $\theta \neq 0, \pi$. The nose-on special case can be considered separately. The tail-on aspect illuminates no part of the cone surface, and the return at that aspect is that of the basal surface. No stationary points exist in the z variation.

Using the fact that:

$$\int_{-h/2}^{h/2} dz(h/2 - z)\mathrm{e}^{i\xi z} = \frac{ih}{\xi}\mathrm{e}^{-ih\xi/2} - \frac{2i}{\xi^2}\sin(h\xi/2)$$

in conjuction with SP, we find:

$$\mathbf{E}_{PO}(\mathbf{R}) = i\nu\frac{\mathrm{e}^{ikR}}{R}B_i\frac{\tan\alpha}{\cos\alpha}\frac{ih}{\xi_0}(\cos\theta\sin\alpha + \sin\theta\cos\alpha) *$$

$$* \,\mathrm{e}^{-ihk\tan\alpha\sin\theta}\left(\mathrm{e}^{-ih\xi_0/2} - \frac{2}{h\xi_0}\sin(h\xi_0/2)\right)\mathrm{e}^{i\pi/4}\sqrt{\frac{2\pi}{kh\tan\alpha\sin\theta}}\hat{\mathbf{x}}$$

where $\xi_0 \equiv 2k(\tan\alpha\sin\theta - \cos\theta)$.

The corresponding RCS is given by:

$$\sigma = \frac{h}{2k}\frac{\alpha}{\sin\theta}\left(\frac{\alpha\cos\theta + \sin\theta}{\alpha\sin\theta - \cos\theta}\right)^2\left(1 - \frac{2}{\eta}\cos\eta\sin\eta + \frac{\sin^2\eta}{\eta^2}\right)$$

where $\eta \equiv h\xi_0/2 \propto h/\lambda$, and $\alpha \ll 1$. SP also requires $kh\tan\alpha\sin\theta \to \infty$. In terms of cone base radius $a \equiv h\tan\alpha$, and $L^2 \equiv a^2 + h^2$:

$$\boxed{\sigma = \frac{aL}{2hk\sin\theta}\left(\frac{a\cos\theta + h\sin\theta}{(La/h)\sin\theta - h\cos\theta}\right)^2\left(1 - \frac{2}{\eta}\cos\eta\sin\eta + \frac{\sin^2\eta}{\eta^2}\right)}$$

CASE 8: Cone: nosetip aspect

The previous case was restricted to aspects away from nose-on. Here we calculate the RCS for this aspect. The entire cone surface is illuminated and:

$$\hat{\mathbf{R}} = \hat{\mathbf{z}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{B}}_i = \hat{\mathbf{y}}$$

so that:

$$\mathbf{E}_{PO}(\mathbf{R}, \text{nose}) = i\nu \frac{e^{ikR}}{R} \frac{\tan\alpha}{\cos\alpha} B_i \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi (h/2 - z)[\sin\alpha\hat{\mathbf{x}} - \cos\alpha\cos\phi\hat{\mathbf{z}}] e^{-2ikz}$$

The term in $\hat{\mathbf{z}}$ integrates to zero over ϕ, and:

$$\mathbf{E}_{PO}(\mathbf{R}, \text{nose}) = \nu \frac{e^{ikR}}{R} \tan^2\alpha B_i \frac{\pi h}{k} \left(e^{ihk} - \frac{\sin(kh)}{kh} \right)$$

The corresponding RCS:

$$\boxed{\sigma = \frac{h^2}{4} \tan^4\alpha \left(1 - \frac{\sin(2hk)}{hk} + \frac{\sin^2(hk)}{(hk)^2} \right)}$$

This formula differs from the oft-quoted result for nosetip scattering of $\lambda^2 \sin^4(\alpha/2)$, which may arise from considering the tip only; we integrated over the length h, so a scaling with h is not surprising. For hk small, we recover a scaling with λ^2.

CASE 9: Cone with spherical nosecap: nose aspect

This is the case of interest for estimating the nose-aspect RCS of a typical missile RV. Estimates for non-nose aspect RCSs can be obtained from the expressions for a cone obtained previously. This calculation is a bit more complicated in that the surface is comprised of two geometrically distinct pieces. The nosecap is naturally expressed in spherical coordinates; the body of the cone in cylindrical coordinates. The surface integral and its integrands are thus decomposed into these two pieces. The nosecap sphere center is put at the origin of coordinates, with the symmetry axis along $\hat{\mathbf{z}}$. The initial polarization is chosen in the $\hat{\mathbf{x}} - \hat{\mathbf{z}}$ plane (see illustration).

$$\hat{\mathbf{R}} = \hat{\mathbf{z}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{B}}_i = \hat{\mathbf{x}}$$

$$dA = r(z)d\phi dz/\cos\alpha \qquad \text{(cone)}$$

$$= r_0^2 \sin\theta d\theta d\phi \qquad \text{(sphere)}$$

$$-\hat{\mathbf{n}} = \cos\phi\cos\alpha\hat{\mathbf{x}} + \sin\phi\cos\alpha\hat{\mathbf{y}} + \sin\alpha\hat{\mathbf{z}} \qquad \text{(cone)}$$

$$= \cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \qquad \text{(sphere)}$$

$$\mathbf{x} = r(z)(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) + z\hat{\mathbf{z}} \qquad \text{(cone)}$$

$$= r_0(\cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) \qquad \text{(sphere)}$$

where $r(z) \equiv (r_0/\sin\alpha - z)\tan\alpha$. The field integral:

$$\mathbf{E}_{PO}(\mathbf{R}) = i\nu\frac{\mathrm{e}^{ikR}}{R}B_i\left[\int_{-h}^{r_0 \sin\alpha} dz \int_0^{2\pi} d\phi \left(\frac{r_0}{\sin\alpha} - z\right)\frac{\tan\alpha}{\cos\alpha}(\sin\phi\cos\alpha\hat{\mathbf{z}} - \sin\alpha\hat{\mathbf{y}})\mathrm{e}^{-2ikz}\right.$$
$$\left. + \int_0^{\pi/2-\alpha} r_0^2\sin\theta d\theta \int_0^{2\pi} d\phi(\sin\phi\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{y}})\mathrm{e}^{-2ikr_0\cos\theta}\right]$$

Both ϕ integrals along $\hat{\mathbf{z}}$ integrate to zero. The other integrals can be done exactly, but there is a proliferation of terms. To reduce terms, consider a case relevant for RVs at UHF: $r_0 \ll h, k^{-1}$. Then:

$$\mathbf{E}_{PO}(\mathbf{R}) \simeq \frac{-\nu\pi}{k^2}\frac{\mathrm{e}^{ikR}}{R}B_i\,[\,\tan^2\alpha\mathrm{e}^{ikh}(\sin(kh) - kh\mathrm{e}^{ikh})$$
$$+\mathrm{e}^{-ikr_0(1+\sin\alpha)}\sin(kr_0 - kr_0\sin\alpha)\,]\,\hat{\mathbf{y}}$$

This reduces to the conical result for $r_0 \to 0$. The associated RCS:

$$\sigma = \pi h^2\left[\tan^2\alpha\mathrm{e}^{ikh}\left(\frac{\sin kh}{kh} - \mathrm{e}^{ikh}\right) + \frac{\mathrm{e}^{-ikr_0(1+\sin\alpha)}}{kh}\sin(kr_0 - kr_0\sin\alpha)\right]^2$$

Since $\tan^2\alpha \ll 1$, this can be simplified:

$$\sigma \simeq \frac{\pi}{k^2}\sin^2(kr_0 - kr_0\sin\alpha)$$

Appendix: Method of Stationary Phase

Consider evaluation of the following integral:

$$I(k) = \int_a^b g(\phi) e^{ikh(\phi)} d\phi$$

In the limit $k \to \infty$, the exponential term oscillates rapidly, and the integral will average to zero over the oscillatory region. Consider 2 cases, depending on the behavior of h:

1) $dh/d\phi \neq 0$:

$$\begin{aligned}
\int_a^b g e^{ikh} d\phi &= \int_a^b \frac{g}{h'} d\left(\frac{e^{ikh}}{ik}\right) \\
&= \int_a^b d\left(\frac{g e^{ikh}}{h' ik}\right) - \int_a^b \frac{e^{ikh}}{ik}\left(\frac{g'}{h'} - \frac{g}{h'^2} h''\right) d\phi \\
&= \frac{g e^{ikh}}{h' ik}\Big|_a^b - \frac{1}{ik}\int_a^b e^{ikh}\left(\frac{g'}{h'} - \frac{g}{h'^2} h''\right) \\
&\propto \frac{1}{k}
\end{aligned}$$

2) $dh/d\phi = 0$; $d^2h/d\phi^2 \neq 0$:

Define the point at which this occurs, the stationary point, ϕ_0. If higher derivatives exist, then near ϕ_0:

$$\begin{aligned}
h &= h(\phi_0) + (\phi - \phi_0) h'(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 h''(\phi_0) + \ldots \\
&\simeq h(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 h''(\phi_0) \\
&\equiv h_0 + \frac{1}{2}(\phi - \phi_0)^2 h_0''
\end{aligned}$$

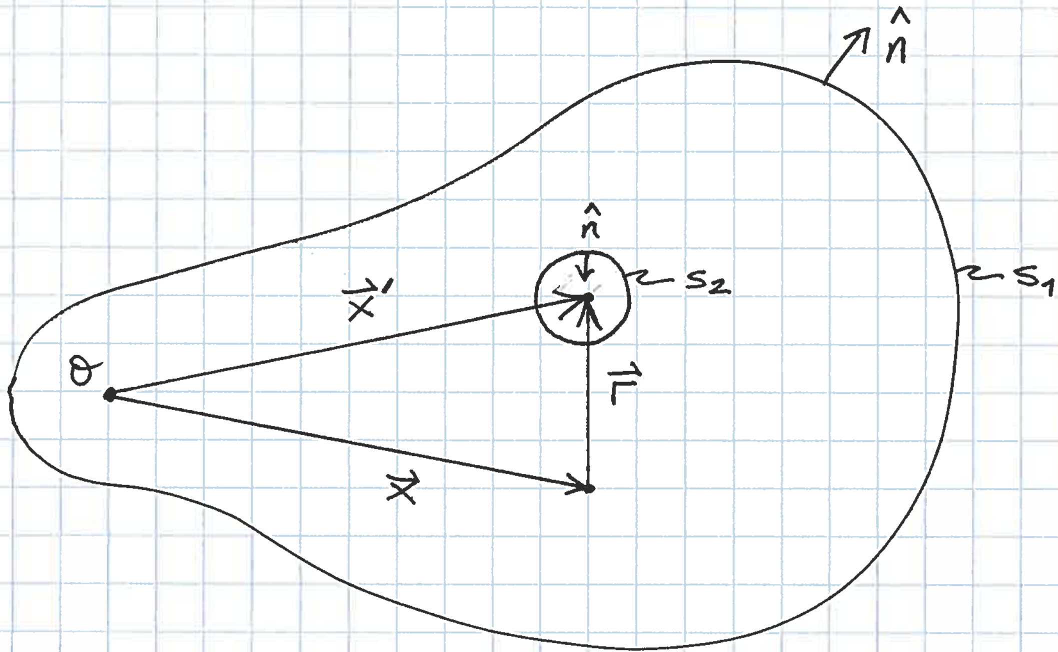
Consider now the integral over a region of width 2ϵ about ϕ_0:

$$\begin{aligned}
I(k) &\simeq \int_{-\epsilon}^{\epsilon} g(\phi) e^{ik(h_0 + \frac{1}{2}[\phi - \phi_0]^2 h_0'')} d\phi \\
&= \int_{-\epsilon k^{1/2}}^{\epsilon k^{1/2}} g(\phi_0 + \psi/k^{1/2}) e^{ikh_0} e^{i\psi^2/2h_0''} \frac{d\psi}{k^{1/2}}, \qquad \psi \equiv k^{1/2}(\phi - \phi_0) \\
&\simeq e^{ikh_0} \frac{g(\phi_0)}{k^{1/2}} \int_{-\infty}^{\infty} e^{i\psi^2/2h_0''} d\psi \\
&= e^{ikh_0} g(\phi_0) \sqrt{\frac{2\pi}{kh_0''}} e^{i\pi/4} \\
&\propto k^{-1/2}
\end{aligned}$$

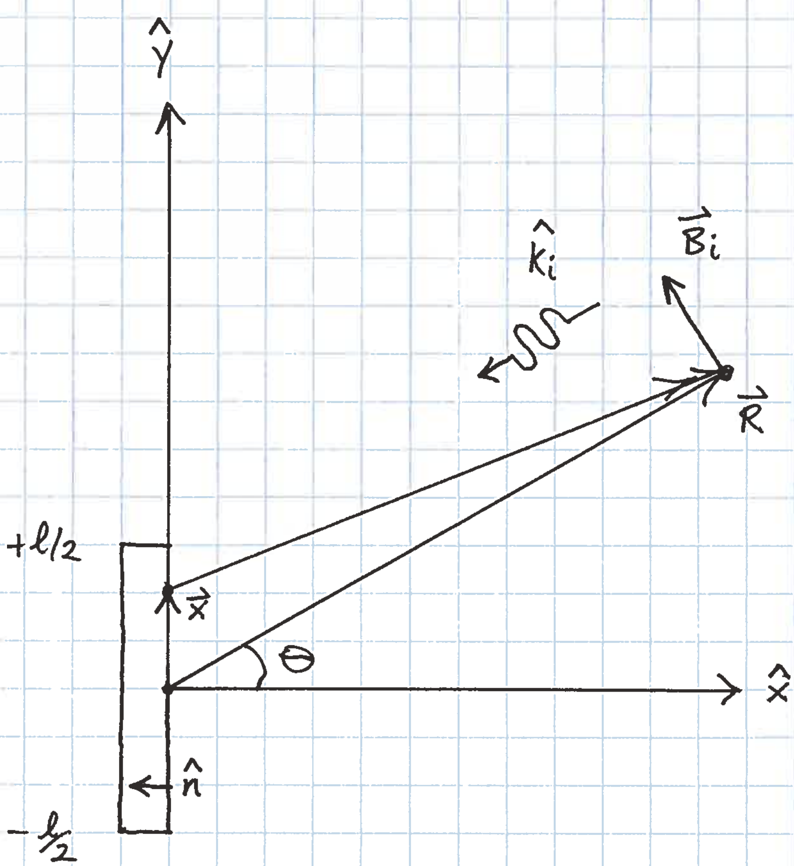
It is assumed that $e^{i\psi^2}$ will vary much faster than $g(\psi)$, and its oscillations will drive the integral to zero everywhere away from ϕ_0, and the limits may be extendend to ∞.

Therefore, near any stationary points ϕ, $I(k) \propto k^{-1/2}$. Otherwise, $I(k) \propto k^{-1}$. As $k \to \infty$, the stationary contributions dominate, and $I(k \to \infty)$ is well-approximated by its value at its stationary point. For multiple stationary points within the integration range, sum over their contributions. We require of course that $h' = 0$ and $h'' \neq 0$.

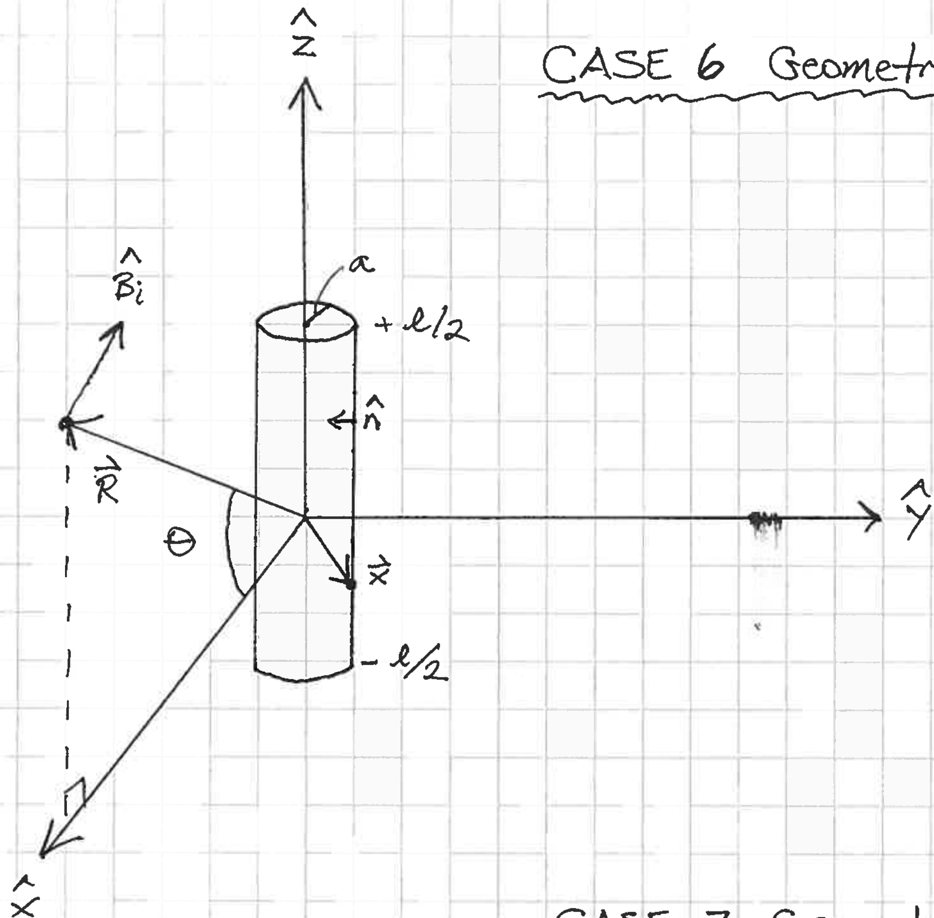
Integral Equation Geometry



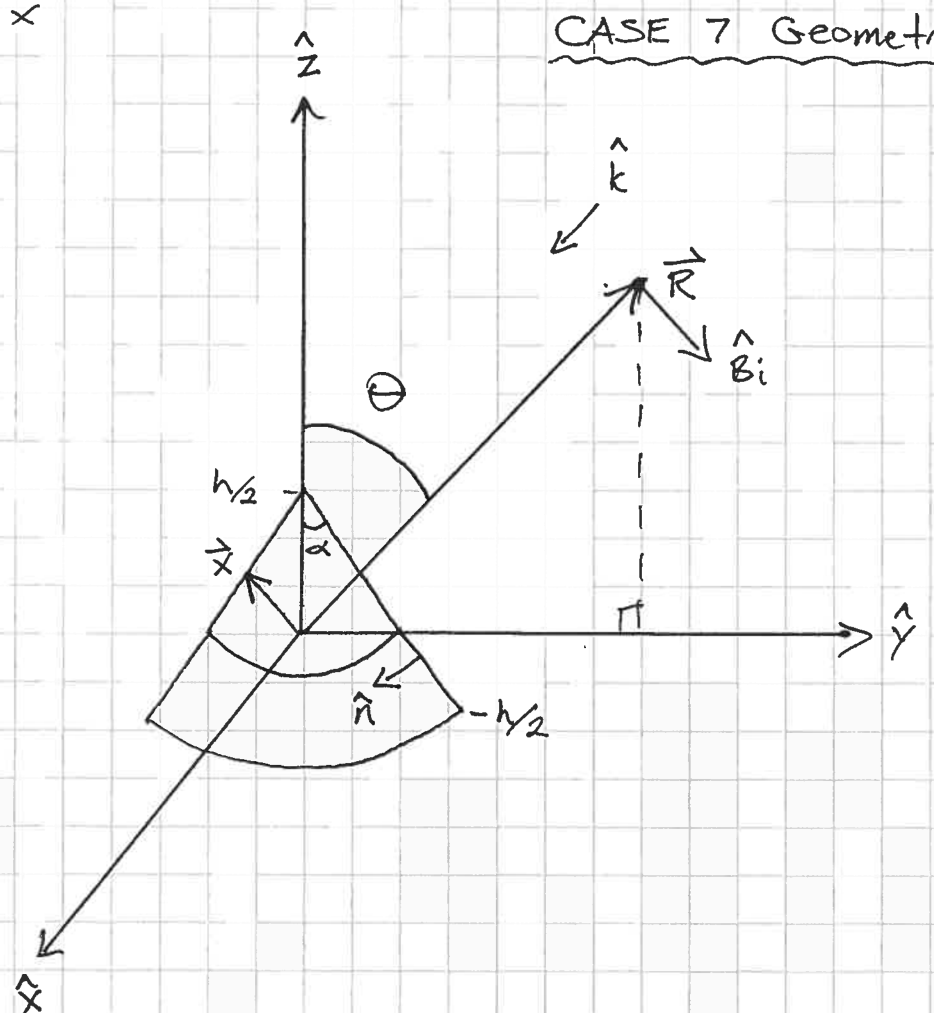
CASE 1 Geometry



CASE 6 Geometry



CASE 7 Geometry



CASE 9 Geometry