Verification of ATMS reflector emissivity calculations

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(Dated: 2 June 2015)

Abstract

This article establishes the formalism to describe the modulation of the microwave signal from the finite emissivity of the rotating reflector onboard the Advanced Technology Microwave Sounder. The standard, polarization-dependent Fresnel equations are used to calculate the polarization-dependent reflectivities from a metal surface for a complex index of refraction characteristic of metals at microwave frequencies. The principle of detailed balance allows us to equate one minus the reflectivity to the emissivity. We depart from standard formulae given in Born & Wolf, *Principles of Optics*, 7th edition, Cambridge University Press, 1999.

This initial draft considers only the reflectivities. The effects of finite reflector emissivity and the temperature of the reflector will be added in an update.

FRESNEL EQUATIONS

The Fresnel equations describe the behavior of radiation, transmitted and reflected, incident on a boundary at which the index of refraction changes. The Fresnel equations distinguish electric field polarization parallel, and perpendicular, to the plane of incidence. At normal incidence, these directions are degenerate. The components of the transmitted electric vector are written $T_{\|}$ and $T_{\perp}$, and those of the reflected electric vector are written $R_{\|}$ and $R_{\perp}$. The incident components of the electric vector are written $A_{\|}$ and $A_{\perp}$. The angle of incidence with respect to the boundary normal is θ_i, and the parameter θ_t describes the transmitted radiation. The incident index of refraction is n_1, and the index changes discontinuously to n_2 across the boundary. The indices of refraction and θ_t are complex numbers.

$$\frac{T_{\|}}{A_{\|}} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} \tag{1}$$

$$\frac{T_{\perp}}{A_{\perp}} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{2}$$

$$\frac{R_{\|}}{A_{\|}} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} \tag{3}$$

$$\frac{R_{\perp}}{A_{\perp}} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{4}$$

These are equations (20) and (21) of Born & Wolf (B&W) Section 1.5. There is an alternative form of the equations (B&W 20a and 21a) that eliminates n_1 and n_2 in terms of θ_t, but the form above is more useful for our purposes.

The angle of reflection will always equal the angle of incidence. The relation between angle of incidence and θ_t is given by Snell's Law, equation (8) in the section of B&W referenced above.

$$n_1 \sin\theta_i = n_2 \sin\theta_t \tag{5}$$

This equation gives complex θ_t in terms of complex n_1 and n_2.

INDEX OF REFRACTION OF METALS AT MICROWAVE FREQUENCIES

A separate report at http://konfluence.org/OM1.pdf provides a derivation for the index of refraction of metals across a wide frequency regime:

$$n^2 = 1 - \frac{i\sigma/\omega\varepsilon_0}{1 + i\omega\tau} \tag{6}$$

σ is the conductivity of the material, ε_0 is the permittivity of free space, ω is the angular frequency of the radiation, and τ is a dissipation parameter that scales with the mean collision time. Born & Wolf provide a derivation of the complex index of refraction from the Maxwell equations equivalent to (6) when $\tau = 0$.

The formula (6) requires consideration of the equations of motion of the material. The complex index of refraction has three characteristic timescales: the frequency $\nu = \omega/2\pi$ of radiation; the conductive timescale σ/ε_0, and the dissipative timescale $1/\tau$. The conductive and dissipative timescales can be combined to provide the plasma frequency ω_p, another characteristic timescale of the material: $\omega_p^2 = \sigma/\varepsilon_0\tau$. Above this frequency, the inertia of the conduction electrons is too great to respond to the radiation, and the conduction electrons become transparent. For most metals, this occurs in the ultraviolet. The plasma frequency depends only on the density of conduction electrons, which is similar for most metals, so ω_p is relatively constant across metals.

The resistivity $1/\sigma$ is 3.5×10^{-8} Ω·m for aluminum; 2.4×10^{-8} Ω·m for gold; and 7×10^{-8} Ω·m for nickel.

The conductive timescale σ/ε_0 is 3.2×10^{18} Hz for aluminum; 4.7×10^{18} Hz for gold; and 1.6×10^{18} Hz for nickel.

The dissipative timescale $1/\tau$ is 6.2×10^{13} Hz for aluminum, and ranges from 10^{14} to 10^{15} Hz for most metals.

The plasma frequency is approximately constant over most metals at between $1 - 2 \times 10^{16}$ Hz.

The ATMS operates between 50 and 180 GHz, or 5×10^{10} to 2×10^{11} Hz. So the radiation frequency $\omega \ll 1/\tau \ll \omega_p \ll \sigma/\varepsilon_0$.

Since $\omega\tau \ll 1$,

$$n^2 = 1 - i\frac{\sigma}{\omega\varepsilon_0}(1 - i\omega\tau) + \mathscr{O}(\omega^2\tau^2)$$

$$
\begin{aligned}
&= 1 - i\frac{\sigma}{\omega\varepsilon_0} - \frac{\sigma\tau}{\epsilon_0} + \mathscr{O}(\omega^2\tau^2) \\
&\simeq -i\frac{\sigma}{\omega\varepsilon_0} = \mathrm{e}^{-\mathrm{i}\pi/2}\frac{\sigma}{\omega\varepsilon_0}
\end{aligned} \tag{7}
$$

where the last approximation follows because $\sigma/\varepsilon_0\omega$ is much larger than any other term.

This implies

$$
n = \pm\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{\sigma}{\omega\varepsilon_0}} \equiv \mathrm{e}^{-\mathrm{i}\pi/4}\kappa(\omega) \tag{8}
$$

This expression for the complex index of refraction of a metal is combined with the Fresnel equations to calculate the emissivity of the metal as a function of polarization.

Since we are considering harmonic solutions to the wave equation for the electric field of the form

$$
E = E_0\mathrm{e}^{\mathrm{i}\omega(\mathrm{t}-\mathrm{nz/c})} \tag{9}
$$

where z is a spatial coordinate, where $n \equiv ck/\omega$, and where k is wavenumber, we can evaluate the skin depth δ for the penetration of radiation from the imaginary part of n:

$$
\delta = \sqrt{\frac{2\varepsilon_0 c^2}{\sigma\omega}} \tag{10}
$$

The skin depth at 50 GHz (lowest frequency and greatest skin depth) is 0.43 micron for aluminum, 0.36 micron for gold, and 0.61 micron for nickel. The ATMS reflector is 0.6 micron of gold on a nickel substrate, over the beryllium reflector. It appears that not much radiation should penetrate to the nickel substrate.

WARM UP CALCULATION:

IMAGINARY INDEX OF REFRACTION AT NORMAL INCIDENCE

Consider $\theta_i = 0$, $n_1 = 1$, and $n_2 = -in_I$, the case of normal incidence onto a purely imaginary index of refraction. The polarization states are degenerate. For $\theta_i = 0$, Snell's law (5) implies $\theta_t = 0$. The Fresnel equations reduce to:

$$
\frac{T_\parallel}{A_\parallel} = \frac{T_\perp}{A_\perp} = \frac{2n_1}{n_1 + n_2} \tag{11}
$$

$$
\frac{R_\parallel}{A_\parallel} = -\frac{R_\perp}{A_\perp} = \frac{n_2 - n_1}{n_1 + n_2} \tag{12}
$$

These are equations (22) and (23) of B&W section 1.5.

The total reflected energy is proportional to the square of the reflected electric field:

$$E_R = RR^* A^2 = \left| \frac{n_2 - n_1}{n_2 + n_1} \right|^2 A^2 \tag{13}$$

which is (37) in B&W section 1.5. For a purely imaginary index of refraction,

$$\left| \frac{R}{A} \right|^2 = \frac{(-in_I - 1)(+in_I - 1)}{(-in_I + 1)(+in_I + 1)} = \frac{1 + n_I^2}{1 + n_I^2} = 1 \tag{14}$$

This implies a purely imaginary index of refraction characterizes a perfect reflector, which is somewhat counter-intuitive, since dissipation arises from the imaginary part of the index of refraction.

WARM-UP CALCULATION:

COMPLEX INDEX OF REFRACTION AT NORMAL INCIDENCE

Consider now $\theta_i = 0$, $n_1 = 1$, and $n_2 = \mathrm{e}^{-\mathrm{i}\pi/4}\kappa(\omega)$. Now we are using the full, complex, low-frequency index of refraction, and κ is a very large number.

$$\begin{aligned} \left| \frac{R}{A} \right|^2 &= \frac{(\mathrm{e}^{-\mathrm{i}\pi/4}\kappa - 1)(\mathrm{e}^{+\mathrm{i}\pi/4}\kappa - 1)}{(\mathrm{e}^{-\mathrm{i}\pi/4}\kappa + 1)(\mathrm{e}^{+\mathrm{i}\pi/4}\kappa + 1)} = \frac{\kappa^2 + 1 - 2^{1/2}\kappa}{\kappa^2 + 1 + 2^{1/2}\kappa} \\ &= \frac{\kappa^2(1 + 1/\kappa^2 - 2^{1/2}/\kappa)}{\kappa^2(1 + 1/\kappa^2 + 2^{1/2}/\kappa)} \\ &= (1 - 2^{1/2}/\kappa)(1 - 2^{1/2}/\kappa) + \mathscr{O}(1/\kappa^2) \\ &\simeq 1 - \frac{2^{3/2}}{\kappa} \end{aligned} \tag{15}$$

where we ignore terms of order $1/\kappa^2$.

Since the fraction of energy absorbed is one minus the fraction reflected, and since the principle of detailed balance requires that absorption equal emission, then we can write down the normal-incidence emissivity:

$$\epsilon_n = \sqrt{\frac{8\omega}{\sigma/\varepsilon_0}} \tag{16}$$

This relation is sometimes called the Hagen-Rubens equation.

Compared with the case of purely imaginary index of refraction, we can see that the real part of the index of refraction is necessary for dissipation.

The values of ϵ_n at 180 GHz (the highest frequency and greatest emissivity) are 1.6×10^{-3} for aluminum; 1.4×10^{-3} for gold; and 2.3×10^{-3} for nickel.

COMPLEX INDEX OF REFRACTION AT 45-DEG INCIDENCE

Consider now $\theta_i = \pi/4$, $n_1 = 1$, and $n_2 = \mathrm{e}^{-\mathrm{i}\pi/4}\kappa(\omega)$. This case should be representative of the ATMS reflector. We have only to substitute the values for these parameters into the Fresnel equations for the reflected electric vector (3), (4), for the components parallel and perpendicular to the plane of incidence.

First note that $\cos\theta_i = 2^{-1/2}$, and (7) applied to Snell's law (5) gives

$$\sin^2\theta_t = \frac{\sin^2\theta_i}{n_2^2} = \frac{\mathrm{e}^{\mathrm{i}\pi/2}}{2\kappa^2} = \frac{i}{2\kappa^2} \tag{17}$$

so that

$$\cos\theta_t = \sqrt{1 - i/2\kappa^2} = 1 + \mathscr{O}(1/\kappa^2) \simeq 1 \tag{18}$$

Substituting now into (3) for the magnitude of the parallel reflectivity, we obtain

$$\begin{aligned}\left|\frac{R_\parallel}{A_\parallel}\right|^2 &= \frac{R_\parallel}{A_\parallel}\frac{R_\parallel{}^*}{A_\parallel} \simeq \frac{(\mathrm{e}^{-\mathrm{i}\pi/4}\kappa\ 2^{-1/2} - 1)(\mathrm{e}^{+\mathrm{i}\pi/4}\kappa 2^{-1/2} - 1)}{(\mathrm{e}^{-\mathrm{i}\pi/4}\kappa\ 2^{-1/2} + 1)(\mathrm{e}^{+\mathrm{i}\pi/4}\kappa 2^{-1/2} + 1)} \\ &= \frac{\kappa^2/2 - \kappa + 1}{\kappa^2/2 + \kappa + 1} = \frac{(\kappa^2/2)(1 - 2/\kappa)}{(\kappa^2/2)(1 + 2/\kappa)} + \mathscr{O}(1/\kappa^2) \\ &= (1 - 2/\kappa)(1 - 2/\kappa) + \mathscr{O}(1/\kappa^2) = 1 - 4/\kappa + \mathscr{O}(1/\kappa^2) \\ &\simeq 1 - 4/\kappa \end{aligned} \tag{19}$$

This equation gives the fraction of incident parallel energy that is reflected.

Likewise, for the fraction of incident perpendicular energy, substitute into (4) for the magnitude of the perpendicular reflectivity:

$$\begin{aligned}\left|\frac{R_\perp}{A_\perp}\right|^2 &= \frac{R_\perp}{A_\perp}\frac{R_\perp{}^*}{A_\perp} \simeq \frac{(2^{-1/2} - \mathrm{e}^{-\mathrm{i}\pi/4}\kappa)(2^{-1/2} - \mathrm{e}^{+\mathrm{i}\pi/4}\kappa)}{(2^{-1/2} + \mathrm{e}^{-\mathrm{i}\pi/4}\kappa)(2^{-1/2} + \mathrm{e}^{+\mathrm{i}\pi/4}\kappa)} \\ &= \frac{\kappa^2 - \kappa + 1/2}{\kappa^2 + \kappa + 1/2} = \frac{\kappa^2(1 - 1/\kappa)}{\kappa^2(1 + 1/\kappa)} + \mathscr{O}(1/\kappa^2) \\ &= (1 - 1/\kappa)(1 - 1/\kappa) + \mathscr{O}(1/\kappa^2) = 1 - 2/\kappa + \mathscr{O}(1/\kappa^2) \\ &\simeq 1 - 2/\kappa \end{aligned} \tag{20}$$

By the same logic that led to (16) from the normal-incidence reflectivities, we can identify the parallel and perpendicular polarization emissivities for a 45° incidence angle in terms of the normal-incidence ϵ_n:

$$\epsilon_n(0^\circ) = 2^{3/2}/\kappa \tag{21}$$

$$\epsilon_\parallel(45^\circ) = 4/\kappa = \sqrt{2}\epsilon_n \tag{22}$$

$$\epsilon_\perp(45^\circ) = 2/\kappa = \epsilon_n/\sqrt{2} \tag{23}$$

APPLICATION TO THE ROTATING REFLECTOR

The ATMS distinguishes polarization directions labelled "V" and "H", nominally for vertical and horizontal. However, due to the transverse nature of radiation, the entire polarization vector viewed at nadir is "horizontal", in that it has no component perpendicular to the earth surface. So the "V" label is somewhat non-physical. But according to the instrument spec, AE-28300, the "V" direction corresponds to the polarization parallel to the plane of incidence at nadir, and the "H" direction corresponds to the polarization perpendicular to the plane of incidence at nadir. In the spacecraft frame, the reflector rotation axis is along the track direction, so the parallel radiation reflected by the reflector at nadir would be perpendicular to the earth surface: "vertical" in the spacecraft frame.

Let us denote the reflector rotation angle as ϕ, and set $\phi = 0$ at nadir. As the reflector rotates, its principle axes (parallel and perpendicular to the plane of incidence) rotate with it. With this, we can express the "V" and "H" components of the received electric field vector in terms of the rotating parallel and perpendicular components:

$$E_V = E_\parallel \cos\phi - E_\perp \sin\phi \tag{24}$$

$$E_H = E_\parallel \sin\phi + E_\perp \cos\phi \tag{25}$$

When we decompose the incident radiation vector into $A_\parallel$ and $A_\perp$, it is important to note that these components do not capture all radiated power at all scan angles. Again, due to the transverse nature of the radiation, the emission from the surface has no component perpendicular to the earth surface. The parallel polarization direction always lies in the plane of the earth surface at all scan angles. However, the perpendicular component is only in the surface plane at nadir. At $\phi \neq 0$, the perpendicular component is geometrically reduced compared to the nadir direction.

So the total polarization of the emissive scene must be resolved into vectors co-planar with the earth surface. The direction parallel to the plane of incidence at the reflector can be chosen as one of the scene basis vectors. But the perpendicular component can not be a basis vector. Let us therefore define source emission polarization bases of parallel ($A_\parallel$ as before), and a surface basis vector A_S orthogonal to the parallel vector. Then, at non-nadir scan angles, we can express:

$$A_\perp = A_S \cos\phi \tag{26}$$

The A_S basis vector is parallel to the scan direction. As the scan angle increases, then $A_\perp$ begins to pick up increasingly larger components perpendicular to the earth surface. If the scan could go out to $\phi = 90°$, then $A_\perp$ would vanish because the source emission can have no component perpendicular to the earth surface. For unpolarized scenes, $A_\parallel^2 = A_S^2$, and so the perpendicular component $A_\perp$ is reduced relative to the parallel component $A_\parallel$ off-nadir.

Now let us express the H and V power received by the instrument, in terms of the reflected polarization components:

$$
\begin{aligned}
|E_V|^2 &= |R_\parallel|^2 \cos^2\phi + |R_\perp|^2 \sin^2\phi \\
&= A_\parallel^2 (1 - 4/\kappa) \cos^2\phi + A_S^2 \cos^2\phi (1 - 2/\kappa) \sin^2\phi \\
&= A_\parallel^2 (1 - \epsilon_\parallel) \cos^2\phi + A_S^2 \cos^2\phi (1 - \epsilon_\perp) \sin^2\phi
\end{aligned} \tag{27}
$$

Likewise,

$$
\begin{aligned}
|E_H|^2 &= |R_\parallel|^2 \sin^2\phi + |R_\perp|^2 \cos^2\phi \\
&= A_\parallel^2 (1 - 4/\kappa) \sin^2\phi + A_S^2 \cos^2\phi (1 - 2/\kappa) \cos^2\phi \\
&= A_\parallel^2 (1 - \epsilon_\parallel) \sin^2\phi + A_S^2 (1 - \epsilon_\perp) \cos^4\phi
\end{aligned} \tag{28}
$$

VIEWING COLD SPACE

As mentioned above, there is no fall-off in scan angle of the perpendicular component when viewing cold space, as there is when viewing the earth. In this case, $|A_\parallel|^2 = |A_\perp|^2 \equiv |A_c|^2$, where we have defined the cold space temperature A_c^2. Let us return, then, to (27) and (28), for the case of cold space:

$$
\begin{aligned}
|E_V|^2 &= A_c^2 (1 - \epsilon_\parallel) \cos^2\phi + A_c^2 (1 - \epsilon_\perp) \sin^2\phi \\
&= A_c^2 [1 - \epsilon_\parallel \cos^2\phi - \epsilon_\perp \sin^2\phi]
\end{aligned} \tag{29}
$$

Likewise,

$$
\begin{aligned}
|E_H|^2 &= A_c^2 (1 - \epsilon_\parallel) \sin^2\phi + A_c^2 (1 - \epsilon_\perp) \cos^2\phi \\
&= A_c^2 [1 - \epsilon_\parallel \sin^2\phi - \epsilon_\perp \cos^2\phi]
\end{aligned} \tag{30}
$$

THERMAL EMISSION FROM THE REFLECTOR

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