Quick-Study Sequence

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prepared by L.L. Williams

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The Fuel Problem

one of two fundamental obstacles to interstellar travel

Travel between the stars, and even between planets around a star, requires propulsion to overcome the effects of gravity. A propulsion source drives motion of the spacecraft with respect to the nearby stars or planets, lifting it out of the local gravitational well.

Chemical rockets are the main propulsion source used in the interplanetary programs. They operate on the principle of throwing something out the back so that the spacecraft is thrown forward. To accelerate a chemical rocket at 1 g half way to the nearest star, and decelerate at 1 g the other half way, would require a fuel tank the size of the moon. Therefore, rockets achieve a speed limited by the fuel they can practically carry. The speed is so small as to require centuries to reach the nearest stars.

Other fuel options include ion beams, microwave sails, and reflecting the blast waves of nuclear explosives. These options do not seem to alter the basic feasibility of chemical rockets, because all these alternative methods yield velocities as small as those of chemical rockets.

To solve the fuel problem would be to provide a limitless source of fuel that could be used to power a spacecraft indefinitely. This would presumably be through some aspect of the universe that is everywhere existent.

It is understood that even with the fuel problem solved, the time-distance problem would remain. The fuel-problem would only allow the continuous acceleration of objects up to speeds approaching that of light, but a solution to the fuel problem alone would not be sufficient to achieve hyper-relativistic travel.

The Time-Distance Problem

one of two fundamental obstacles to interstellar travel

The Time-Distance Problem is the really profound problem of interstellar travel. Even if the fuel problem were solved, and we could accelerate freely in any gravitational field, accelerating even up to the speed of light, our civilization could still not explore the stars. This is because the severe effects of time dilation would isolate any emissary or probe in time, very much like a Planet-of-the-Apes scenario. Our astronauts could see the center of the galaxy, but they would return to the far future of their planet, and their civilation would be gone.

It is best to approach this problem mathematically, so that we can be awed by the fundamental simplicity of our obstacle. In essence, the time-distance problem is inherent to the structure of space and time.

We understand space and time to be joined together in a spacetime continuum. Moreover, "distance" between any two events in spacetime is invariant with respect to the state of motion.

$$c^2 d\tau^2 \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

where t is a time coordinate, x, y, and z are spatial coordinates, τ is called the proper time, and c is the speed of light. Greek superscripts denote the 4 components of space and time.

In terms of 4-velocity $U^\mu \equiv dx^\mu/d\tau$, the invariant interval (1) implies

$$\eta_{\mu\nu} U^\mu U^\nu = c^2 \tag{2}$$

This equation implies

$$\eta_{\mu\nu} U^\mu \frac{dU^\nu}{d\tau} = 0 \tag{3}$$

It's a general rule of motion in spacetime – we haven't done any physics yet – that the 4-velocity and the 4-acceleration are orthogonal vectors.

Consider a spaceship moving in one direction. In the rest frame of the moving spaceship, where the coordinate time is the proper time, $U^\mu_{ship} = (c, 0, 0, 0)$, and $U^2 = c^2$ in all frames, consistent with equation (2).

Consider now the simple case of constant acceleration in the x direction. One viable trajectory to the stars would be to accelerate at 1 g half way, and decelerate at 1 g the other half way; thereby, maintaining artifical gravity for the astronauts. From (3) we deduce that in the rest frame of the spaceship, $(dU^\mu/d\tau)_{ship} = (0, a, 0, 0)$, where a is the acceleration or effective gravity measured in the frame of the spaceship. It follows that $(dU/d\tau)^2 = -a^2$ in all frames.

Therefore we obtain the two equations in two unknowns $t(\tau)$ and $x(\tau)$:

$$c^2 = \eta_{\mu\nu} U^\mu U^\nu = c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 \tag{4}$$

$$-a^2 = \eta_{\mu\nu} \frac{dU^\mu}{d\tau}\frac{dU^\nu}{d\tau} = c^2 \left(\frac{d^2t}{d\tau^2}\right)^2 - \left(\frac{d^2x}{d\tau^2}\right)^2 \tag{5}$$

The solutions to this pair of equations are:

$$t(\tau) = \frac{c}{a} \sinh\left(\frac{a\tau}{c}\right) \tag{6}$$

$$x(\tau) = \frac{c^2}{a} \cosh\left(\frac{a\tau}{c}\right) \tag{7}$$

These simple equations tell us how time and distance pass for a ship under constant acceleration. The parameter τ is the time coordinate on board the ship. And we haven't even done any physics!

Equation (6) tells us that for acceleration a at 1 g, and onboard elapsed time τ of 5 years, 69 years would pass on earth, far longer than a typical space program lifecycle. Yet that would scarely allow the astronauts to reach the nearest stars.

Equations (6) and (7) can be combined to demonstrate the limiting speed of light:

$$\frac{dx}{dt} = \frac{at}{\sqrt{1 + a^2t^2/c^2}} \leq c \tag{8}$$

The time-distance problem is that motion in space time introduces severe time dilation effects that completely cut off contact between the earth and any distant explorers. And we haven't even done any physics!

The warp drive solution to the Einstein equations by Alcubierre, and the worm hole solutions to the Einstein equations studied by Kip Thorne, would get around this speed limit. By bending spacetime itself, effective hyper-relativistic speeds can in principle be achieved; but not in practice. Not yet, anyway. Another way around the time-distance problem might involve hyperdimensions. But the time-distance problem is quite profound and fundamental.

Warp Drives and Wormholes: Good News, Bad News

The good news is that general relativity *appears* to allow in principle the traversal of distances faster than light speed, and the solution of the Time-Distance Problem.

From equation (8) of the Time-Distance Problem summary, you might infer that the limiting speed will be the coefficient of the time component in the metric. In flat space, $\eta_{tt} = c^2$. In the curved spacetime around a black hole, the Schwarzschild metric gives the time-time component as

$$g_{tt} = c^2 \left(1 - \frac{2GM}{rc^2}\right)$$

We conclude the speed of light gets smaller as one approaches a star, and note that the limiting speed c need not be the limit everywhere in spacetime. Since mass affects the speed of light, one might hope to control the shape of space and time to overcome the time-distance problem. This is where warp drives and wormholes come in.

The Einstein equations describe black holes, warp drives, and worm holes. They are written:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

These equations are in the 10 unknowns $g_{\mu\nu}$. The quantities in $R_{\mu\nu}$ and R are second order derivatives of the $g_{\mu\nu}$.

The quantity $T_{\mu\nu}$ is the stress energy tensor, with units of energy density. Some are surprised to learn that there is no formula or prescription for choosing $T_{\mu\nu}$, but there are many standard forms for various situations.

The Newtonian analogue of (1) is

$$\nabla^2 \phi = 4\pi G \rho$$

In either case, a second derivative is related to a mass/energy density, with the coupling constant G. Therefore, curvature of space is quantified by the magnitude of G. The units of the LHS of (1) are $1/l^2$. Therefore (1) gives a curvature scale:

$$L_\rho \sim \sqrt{c^2/G\rho} \tag{2}$$

We may reasonably hope to transcend the light barrier by building a wormhole or bubble of some sort, and (2) allows us to estimate how much mass density is needed for a given size warp or hole. A 15-meter bubble requires a mass density of 1 earth mass per cubic *meter*. This seems far beyond our means. Because the gravitational constant is so small, astronomically large amounts of mass density are needed to curve space appreciably.

If we substitute $\rho \sim M/L^3$, then we recover the Schwarzschild radius of $R_S \sim MG/c^2$. The Scwharzschild radius of the earth is 4 millimeters. Even cobbling together a mass the size of earth could only give you a warp or wormhole big enough to transport a flea.

But if that is not enough to depress you, there is more.

The equation (1) universally describes an attractive force of gravity. Einstein did not anticipate any sign change on the RHS of (1). But to build a workable bubble or wormhole, the mass energy must push out like antigravity, and we hope to somehow insert a minus sign in (1). This is sometimes called negative energy. Standard physics only tells us how a black hole can eat you, but we have no idea how to get one to barf you back out, like Jonah from the whale.

The problem is we have never seen or measured negative energy. Perhaps its best known example is the quantum vacuum, but the measured value of the vacuum energy in cosmology is orders of magnitude smaller than we expect from quantum theory. So we really don't even understand the quantum vacuum, let alone other forms of exotic energy.

Wormholes and warp drives are only achievable with astronomical quantities of something unknown to science.

Maxwellian Gravity

Reference: *Spacetime and Geometry*, by Sean Carroll, 2004, section 7.2

General Relativity (GR) is our theory of gravity, space, and time. The theory has been confirmed experimentally in many different ways. Yet the equations are enormously complex. Specifically, they are non-linear. Unlike the Maxwell equations, there are no systematic techniques to obtain solutions to the Einstein field equations. All solutions – such as the Robertson-Walker metric, the Schwarzschild metric, the Kerr metric, etc – are guessed and then verified. This means there may well be complexity in the field equations that we have not yet discovered.

Therefore, linear solutions have been sought to the equations since Einstein. To linearize GR, one expresses the full spacetime metric $g_{\mu\nu}$ into a perturbation $h_{\mu\nu}$ about the flat space metric $\eta_{\mu\nu}$, such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $h_{\mu\nu} \ll 1$. This expression is plugged into the equations of GR and terms are only kept to linear order in $h_{\mu\nu}$.

Carroll suggests decomposing the components of $h_{\mu\nu}$ according to their spatial transformation properties. Then $h_{00} \equiv \phi$ transforms as a scalar, and $h_{0i} \equiv \mathbf{A}$ transforms as a vector. The remaining components are the 3x3 matrix h_{ij}. This decomposition is entirely analogous to decomposing the electromagnetic field strength tensor into electric and magnetic field vectors.

For a particle of energy E and velocity $\mathbf{v}$ moving in this gravitational field, its energy equation is (Carroll 7.23)

$$\frac{dE}{dt} = -E\left[\frac{\partial\phi}{\partial t} + 2\mathbf{v}\cdot\nabla\phi - \mathcal{O}(v^2)\right]$$

Now define gravito-electric field $\mathbf{G}$ and gravito-magnetic field $\mathbf{H}$:

$$\mathbf{G} \equiv -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$$
$$\mathbf{H} \equiv \nabla\times\mathbf{A}$$

Then the particle spatial momentum $\mathbf{p}$ is described by (Carroll 7.26):

$$\frac{d\mathbf{p}}{dt} = E\left[\mathbf{G} + \mathbf{v}\times\mathbf{H} - \mathcal{O}(\partial h_{ij}/\partial t, v^2)\right]$$

These are clearly very similar to the Lorentz force law of electromagnetism, and one can infer that GR includes magnetic-type gravitational forces. Unlike the Maxwell equations, however, we also have additional, non-linear terms.

Interestingly, Carroll goes on to show how the only components with true degrees of freedom are the spatial components h_{ij}. The suggestive gravito-electric and gravito-magnetic fields are fixed by the stress energy tensor and the h_{ij}. Carroll goes on to choose a particular gauge for the $h_{\mu\nu}$, and to show the h_{ij} is the piece that contains gravitational radiation.

Inertia from Gravity: Insights of Sciama

Reference: *On the Origin of Inertia*, D.W. Sciama, MNRAS, 113, 1953

The fuel problem (see EP Quick Study I) can also be seen as a problem of inertia. If it weren't for inertia, we would not need any fuel to push a spaceship. The resistance to acceleration comes from inertia.

We may expect there to be some link between gravity and inertia, since they have the same mass parameter. While Einstein built General Relavitity on the equivalence of gravitation and inertia, the origin of inertia is still debated. Insight into the origin of inertia came from a simple argument by Sciama. This insight has formed a cornerstone of Woodward's Mach Effect theory.

Sciama quantifies inertia by assuming gravity is a vector field and using the well-known mathematics of electrodynamics to quickly derive its inertial implications. We know, and Sciama knew, that gravity is a tensor field, but it is a reasonable approximation because Sciama's result only requires that gravity be at least as complex as a vector field. Einsteins equations of gravity can be linearized, and they do indeed show a Maxwellian character (see EP Quick Study IV). There are of course the gravito-electric effects one expects for Newtonian gravity, but other gravito-magnetic effects that Newton would not have recognized. These latter effects are more conventionally known as frame dragging.

So Sciama starts with a Maxwellian gravitational 4-vector potential $(\phi, \mathbf{A})$. The scalar potential has the usual mathematical form, but in terms of mass density ρ instead of charge density, and in terms of the gravitational constant G instead of the dielectric constant:

$$\phi = -G \int \frac{\rho}{r} dV$$

. The integral is taken over the whole universe, but we know the universe is undergoing a Hubble expansion of speed $v_H = Hr$, where H is the Hubble parameter. The current value of $H_0 \simeq 70$ km/s per megaparsec. It is presumed no part of the universe receding faster than the speed of light can influence the local scalar field of gravity, so the integral is cut off at a distance $r_H = c/H$. The value of H changes over time, and the horizon distance is the farthest object to have emitted light just now reaching us. The horizon distance depends on the energy content of the universe, but it scales with the instantaneous horizon distance $r_H = c/H \sim c/H_0$. Therefore we can approximate the scalar potential of the universe:

$$\phi \simeq -G \int_0^{r_H} \frac{\rho}{r} 4\pi r^2 dr = -2\pi G \rho_U \left(\frac{c}{H_0}\right)^2$$

where we have approximated the mass density of the universe as a constant ρ_U. This equation already contains a compelling feature. If taken at infinity, the integral would diverge quadratically. This means the inertia here is dominated by matter in the distant universe.

We construct the usual spatial vector potential in terms of a mass current $\mathbf{J}$:

$$\mathbf{A} = -\frac{G}{c^2}\int \frac{\mathbf{J}}{r} dV$$

At this point, Sciama invokes Mach's principle to presume that there is a rest frame for the universe, and in that frame the current must be zero. But for an object in motion with respect to the rest frame of the universe, the universe appears to be in motion and the object at rest. Therefore an apparent current arises in the universe from the motion of the object. In this case, the current of the universe due to the motion $\mathbf{v}$ of the object is $\mathbf{J} = \rho\mathbf{v}$. The gravitational vector potential of the universe is then:

$$\mathbf{A}_U = -\frac{G}{c^2}\int_0^{r_H} \frac{\rho\mathbf{v}}{r} 4\pi r^2 dr = \phi_U \frac{\mathbf{v}}{c^2}$$

Construct the usual Maxwellian gravito-electric force from the potentials:

$$\mathbf{f} = -\nabla\phi_U - \frac{\partial \mathbf{A}_U}{\partial t} = 2\pi G \frac{\rho_U}{H_0^2}\frac{\partial \mathbf{v}}{\partial t}$$

Now the identification of inertia is complete, if the coefficient leading the derivative of the velocity is unity. Remarkably – and unknown in Sciama's time – cosmology tells us it is.

The Friedmann equation is the workhorse of modern cosmology. It relates the expansion of the universe $H(t)$ to curvature κ and energy density $\epsilon(t)$:

$$H^2(t) = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2 a^2(t)}$$

For a flat universe, $\kappa = 0$ and the energy density of the universe is the critical density:

$$H^2(t) = \frac{8\pi G}{3c^2}\epsilon_c(t)$$

In fact, modern cosmology tells us that the universe is flat and the equation above relates the Hubble parameter to the energy density of the universe. If we set $\epsilon_c = \rho_U c^2$ with the understanding that ρ_U includes all the gravitating mass-energy in the universe, we find

$$H_0^2 = \frac{8\pi G}{3}\rho_U$$

and therefore

$$\mathbf{f} = \frac{3}{4}\frac{\partial \mathbf{v}}{\partial t}$$

The coefficient is close enough to unity that we would have to revisit our approximation to GR, our integration limit, and our detailed model of the expansion of the universe, to definitively rule out gravity as the origin of inertia. Modern cosmology and the flat universe appears to reinforce the conclusion that inertia can be accounted for by the gravitational influence of the universe.

A closely related result of the flat universe is that the gravitational potential energy of every particle in the universe exactly equals its rest mass energy. That is, $\phi = c^2$. Therefore, the speed of light is set by the gravitational potential of the universe.

Woodward's Mach Equation

This is a summary of Woodward's own derivation of the Mach equation, as given in his book *Making Starships and Stargates*. It is compressed to facilitate comparison and analysis.

Woodward starts with the observation from Sciama, 1953, that gravitational forces from the rest of the universe can account for inertia. The inertial force is per unit mass, like a gravitational field; call it $\mathbf{f}$.

A relativistic description of force involves the proper time derivative of the 4-momentum:

$$F^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{dp^0}{d\tau}, \frac{d\mathbf{p}}{d\tau} \right) \tag{1}$$

This is a 4-vector expression. Greek indices range over the 4 coordinates of spacetime. The time component of the 4-vector is noted with a 0 index, and the 3 spatial components are written as a vector in bold-face. The time component of the 4-momentum is the energy. Therefore a 4-vector that corresponds to the inertial force $\mathbf{f}$ per unit mass will have a time-component equal to the work done on the object by the force, per unit mass:

$$\frac{F^\mu}{m} = f^\mu = \left(\frac{1}{m} \frac{dp^0}{d\tau}, \mathbf{f} \right) \tag{2}$$

Woodward elects to investigate the 4-divergence of the relativistic inertial force field:

$$\nabla_\mu f^\mu = \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{m} \frac{dp^0}{d\tau} \right) - \nabla \cdot \mathbf{f} \tag{3}$$

Woodward posits that the divergence (3) of the inertial force must correspond to a field equation. He sets the right hand side equal to some unspecified source $4\pi Q$, as might be expected for such an equation. He converts the particle energy into an energy density E, to conform to the expected units in a field equation. And he considers non-relativistic speeds, so that the proper time derivative becomes a simple time derivative, and the energy density is simply the mass density time the speed of light squared, c^2:

$$\frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\partial E}{\partial t} \right) - \nabla \cdot \mathbf{f} = 4\pi Q \tag{4}$$

The first term on the left side expands into 2 terms, differing by a minus sign due to the derivative of the inverse mass density. Now Woodward makes 3 simultaneous ansatzes, one for each term of equation (4):

1. The source term is the usual source term of Newtons law of gravity: $Q \to G\rho$, the product of mass density and Newton's gravitational constant

2. The inertial reaction force $\mathbf{f} \to -\nabla\phi(\mathbf{x}, t)$, the gradient of the gravitational potential in Newton's law of gravity

3. The energy density $E \to \rho(\mathbf{x}, t)\phi(\mathbf{x}, t)$. This is the Mach hypothesis: that the energy of a body depends on the gravitational potential. Equivalently, the speed of light squared is the same as the value of the scalar potential, $c^2 \to \phi(\mathbf{x}, t)$.

With these assumptions and a prescription for where to make the substitutions, Woodward calculates his Mach effect equation, which is a modification of Newtons field equation:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 4\pi G\rho + \frac{\phi}{\rho c^2}\frac{\partial^2\rho}{\partial t^2} - \left(\frac{\phi}{\rho c^2}\right)^2\left(\frac{\partial\rho}{\partial t}\right)^2 - \frac{1}{c^4}\left(\frac{\partial\phi}{\partial t}\right)^2 \tag{5}$$

The basic feature of Woodwards equation (5) is time-derivatives of the field acting as a source, and these sources are scale free: they dont involve the gravitational constant. So they are relatively larger than the conventional source term, the first term on the right side. They also have the negative-definite terms that Woodward would suggest for creating warp drives or wormholes.

As Woodward notes, equation (5) does not tell us anything about how the mass time derivatives come about. Naively it would appear to say a change in internal energy of any sort could bring about this effect. But Woodward hearkens back to the original assumption, we could call it Woodwards zeroth ansatz, that inertial reaction forces arise from the rest of the universe resisting the acceleration of a body. If the body is not accelerated, no inertial reaction force is raised.

Woodwards 4th ansatz is that the change in internal energy scales with the acceleration that raises the inertial reaction force. So he then equates the inertial reaction force with the bulk acceleration of the object:

$$\Delta E = m\mathbf{f} \cdot \Delta\mathbf{s} = m\mathbf{a} \cdot \Delta\mathbf{s} \tag{6}$$

The quantity $\Delta\mathbf{s}$ is a parameterization of the work done by the inertial reaction force, with units of length. It is understood that this does *not* correspond to the bulk displacement of the object, but rather, to some internal dissipation. This gives

$$\frac{\partial E}{\partial t} = m\mathbf{a} \cdot \frac{\partial\mathbf{s}}{\partial t} \tag{7}$$

and

$$\frac{\partial^2 E}{\partial t^2} = m\mathbf{a} \cdot \frac{\partial^2 \mathbf{s}}{\partial t^2} + \frac{\partial \mathbf{s}}{\partial t} \cdot \left(m\frac{\partial \mathbf{a}}{\partial t} + \mathbf{a}\frac{\partial m}{\partial t} \right) \tag{8}$$

Finally, Woodward hypothesizes that the parameterization of dissipation $\partial \mathbf{s}/\partial t$ must scale with the bulk velocity $\mathbf{v}$, so that $\partial \mathbf{s}/\partial t = \eta \mathbf{v}$. Then, since $\mathbf{a} = \partial \mathbf{v}/\partial t$, Woodward finds (3.7) of his book:

$$\frac{\partial^2 E}{\partial t^2} = \eta m a^2 + \eta \mathbf{v} \cdot \left(m\frac{\partial \mathbf{a}}{\partial t} + \mathbf{a}\frac{\partial m}{\partial t} \right) \tag{9}$$

At successive points in Woodwards development, he drops the second term as much smaller than the first term in the equation above, and also drops the term quadratic in $\partial m/\partial t$ in (5). Adopting these approximations now, and putting all this together, we obtain the approximate equation for the mass fluctuation:

$$\delta m = V\delta\rho = \frac{\eta}{4\pi G}\frac{Va^2}{c^2} = \frac{\eta}{4\pi G}\frac{ma^2}{\rho c^2} \tag{10}$$

which is equation (5.9) from Woodwards book, and where the mass fluctuation is defined in terms of the standard Newtonian expression:

$$\nabla^2 \phi = 4\pi G(\rho + \delta\rho)$$

This is apparently the mass fluctuation that is produced from accelerating objects, and which Woodward would hope to engineer into various designs that time the mass fluctuations with internal constituent motions of the thruster to produce a net impulse.

Note that the gravitational constant enters inversely, which presumably results in very large effects.